King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics
MATH 302
EXAM I
2015-2016 (151)

Wednesday, October 14, 2015 **Allowed Time: 2 Hours**

Name: Correction

ID Number: ______________________ **Serial Number:** ______________

Section Number: ____________ **Instructor's Name:** ______________

Instructions:

1. Write neatly and legibly. You may lose points for messy work.

2. **Show all your work.** No points for answers without justification.

3. **Programmable Calculators and Mobiles are not allowed.**

4. Make sure that you have 5 different problems (5 pages + cover page).

Problem No.	Points	Maximum Points
1		10
2		8
3		10
4		7
5		15
Total:		50

Q1. (a) Let λ be a real number and **A** be an $n \times n$ matrix. ***Show*** that $\boldsymbol{E}_\lambda = \{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}\}$ is a ***subspace*** of $\mathbb{R}^n$. ($\boldsymbol{E}_\lambda$ is called the eigenspace of **A** corresponding to the eigenvalue λ.)

- $0 \in E_\lambda$ since $A(0) = \lambda(0) = 0$
- let $x, y \in E_\lambda \iff Ax = \lambda x$ and $Ay = \lambda y$

$A(x+y) = Ax + Ay = \lambda x + \lambda y = \lambda(x+y)$

So $x+y \in E_\lambda$

- let $k \in \mathbb{R}$ and $x \in E_\lambda$. Then

$A(kx) = kAx = k(\lambda x) = \lambda(kx) \Rightarrow kx \in E_\lambda$

Therefore E_λ is a subspace of $\mathbb{R}^n$.

(b) If λ is a real eigenvalue with multiplicity m and **A** is a diagonalizable matrix. Find dimension of the eigenspace $\boldsymbol{E}_\lambda$. Give a reason for your answer.

If A is diagonalizable, then λ must have m linearly independent eigenvectors $E_1, \dots, E_m$ such that $AE_i = \lambda E_i$, $\forall i = 1, \dots, m$.

$\{E_1, \dots, E_m\}$ forms a basis to E_λ.

So $\dim E_\lambda = m$.

Q2. In R^3, let $\mathbf{u_1} = \langle 1,-1,0\rangle$, $\mathbf{u_2} = \langle 2,0,1\rangle$, $\mathbf{u_3} = \langle 1,1,\alpha\rangle$.

(a) Find all values of α for which $\{\mathbf{u_1},\mathbf{u_2},\mathbf{u_3}\}$ forms a basis in R^3.

$\{u_1, u_2, u_3\}$ forms a basis iff u_1, u_2, u_3 are linearly independent. So we may use the determinant or Gauss' elimination

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & \alpha \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & \alpha \end{pmatrix} \xrightarrow[R_3/2]{R_2/2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1/2 \\ 0 & 1 & \alpha/2 \end{pmatrix}$$

$$\xrightarrow[R_3 - R_2]{R_1 + R_2} \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & \frac{\alpha-1}{2} \end{pmatrix}$$

So u_1, u_2, u_3 are linearly indep iff $\boxed{\alpha \neq 1}$

(b) Take $\alpha = 2$, let $\mathbf{a} = \langle 5,-3,4\rangle$. Express the vector $\boldsymbol{a}$ as linear combination of the vectors $\mathbf{u_1},\mathbf{u_2},\mathbf{u_3}$. That is find $\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}$ such that $\mathbf{a} = \mathbf{x_1}\mathbf{u_1} + \mathbf{x_2}\mathbf{u_2} + \mathbf{x_3}\mathbf{u_3}$.

$$a = \langle 5,-3,4\rangle = x_1\langle 1,-1,0\rangle + x_2\langle 2,0,1\rangle + x_3\langle 1,1,2\rangle$$

$$= \langle x_1 + 2x_2 + x_3,\ -x_1 + x_3,\ x_2 + 2x_3\rangle$$

So, we get
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 5 \\ -x_1 \qquad + x_3 &= -3 \\ x_2 + 2x_3 &= 4 \end{aligned}$$

We have to solve the system

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ -1 & 0 & 1 & -3 \\ 0 & 1 & 2 & 4 \end{array}\right) \xrightarrow{R_2 + R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & 2 & 4 \end{array}\right) \xrightarrow{R_2/2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right) \xrightarrow[R_3 - R_2]{R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow[R_2 - R_3]{R_1 + R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$$

$$a = 6u_1 - 2u_2 + 3u_3$$

Q3. (a) Find matrix $\mathbf{A}^{-1}$ if $\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3+R_2]{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_2+R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

So $A^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

(**b**) Solve the system $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A}^{-1}$ is the matrix found in (a),

$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

$$X = A^{-1}B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 3 \end{pmatrix}.$$

Q4. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & -3 \\ 1 & 1 & -3 & 0 \\ 3 & 1 & -1 & -6 \end{pmatrix}$.

(a) Find the rank of **A**.

$$\begin{pmatrix} 1 & 0 & 1 & -3 \\ 1 & 1 & -3 & 0 \\ 3 & 1 & -1 & -6 \end{pmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - R_3} \begin{pmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -4 & 3 \\ 0 & 1 & -4 & 3 \end{pmatrix} \xrightarrow{R_3 - R_2}$$

$$\begin{pmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{Rank } A = 2$$

(b) Determine whether the vectors

$\mathbf{u_1} = \langle 1,0,1,-3 \rangle$, $\mathbf{u_2} = \langle 1,1,-3,0 \rangle$, $\mathbf{u_3} = \langle 3,1,-1,-6 \rangle$

are linearly independent or linearly dependent. Give reason(s) for your answer.

u_1, u_2, u_3 are the rows of A.

rank $A = 2 < 3$ ~~...~~. So u_1, u_2, u_3 are linearly dependent

(c) How many *parameters* does the solution of the system $\mathbf{AX} = \mathbf{0}$ have?

$$AX = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since rank $A = 2$. Then the solution with dependent on 2 ~~other~~ parameters

That is

$$x_1 + x_3 - 3x_4 = 0$$

$$x_2 - 4x_3 + 3x_4 = 0$$

$$\Rightarrow \quad x_1 = -x_3 + 3x_4$$

$$x_2 = 4x_3 - 3x_4$$

$$\therefore \quad X = \begin{pmatrix} -x_3 + 3x_4 \\ 4x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

Q5. Let $\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Explain why **A** is ***diagonalizable***.

A is symmetric, so it is diagonalizable.

(b) Verify that eigenvalues of **A** are $\lambda_1 = -1$ and $\lambda_2 = \lambda_3 = 1$.

$$\begin{vmatrix} -\lambda & -1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2-1)$$

$$= (1-\lambda)(1-\lambda)(1+\lambda) = 0 \Rightarrow \lambda = -1, 1, 1.$$

(c) Find the ***matrix*** **P** that diagonalizes **A** and find the diagonal matrix $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$.

$\boxed{\lambda = -1}$ $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{R_2+R_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$\Rightarrow \quad 2x_3 = 0 \Leftrightarrow x_3 = 0$ and $x_1 = x_2$. So take $E_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

$\boxed{\lambda = 1}$ $\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$\Rightarrow \quad x_1 + x_2 = 0 \Rightarrow x_2 = -x_1$

$\Rightarrow \quad X = \begin{pmatrix} x_1 \\ -x_1 \\ x_3 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. So take $E_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Thus $P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ diagonalizes A such that

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(d) Compute $\mathbf{A}^{40}$.

$$A^{40} = PD^{40}P^{-1} = PIP^{-1} = PP^{-1} = I$$

where $D^{40} = \begin{pmatrix} (-1)^{40} & 0 & 0 \\ 0 & 1^{40} & 0 \\ 0 & 0 & 1^{40} \end{pmatrix} = I$.