King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math-280, Term-151
Final Exam, Time Allowed: 3 hours

Name: **ID:**

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Question	Score	Total Mark
1		20
2		20
3		20
4		20
5		20
6		20
7		20
8		20
9		20
10		20
TOTAL		200

Question 1: A linear operator L is defined on P_3 as

$L(p(x)) = p(0)x + p(1)$. Find

a) the kernel of L.

b) the range of L.

Question 2: Let $L: V \to W$ be a linear transformation from the vector space V to the vector space W .

a) Show that L is one-to-one if and only if $\ker(L) = \{0\}$.

b) Is the linear transformation $L: \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$L(x) = (x_1, 0, 2x_3)^T$, for all $x = (x_1, x_2 . x_3)^T \in \mathbb{R}^3$

one-to-one. Explain.

Question 3: Let L be a linear operator defined on $\mathbb{R}^2$ as follows

$$L(x) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}, \qquad \text{for all } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2.$$

If $B_1 = \{u_1, u_2\}$ and $B_2 = \{v_1, v_2\}$ are two ordered bases for $\mathbb{R}^2$, with

$$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

then find

a) the transition matrix S corresponding to the change of basis from B_1 to B_2.

b) the matrix A representing L with respect to B_2.

Question 4: Let

$$S = \left\{ x \in \mathbb{R}^4 : x = \alpha \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \end{bmatrix} \right\}$$

Find a basis for $S^{\perp}$.

Question 5: Define an inner product on P_5 as follows

$$\langle p,q\rangle = \sum_{i=1}^{5} p(x_i)q(x_i)$$

where $p,q \in P_5$ and $x_i = \frac{i-3}{2}, \quad i = 1,2,3,4,5$. Compute

a) $\langle x, x^2\rangle$

b) The distance between x and x^2.

Question 6: Let $\{x_1, x_2, x_3\}$ be an orthonormal basis for an inner product space V and let

$$u = x_1 + 2x_2 + 2x_3 \qquad \text{and} \qquad v = x_1 + 7x_3$$

Find the value of the following

a) $\langle u, v \rangle$

b) $\|u\|$ and $\|v\|$

c) The angle θ between u and v.

Question 7: Let $A = \begin{bmatrix} 3 & -1 \\ 4 & 2 \\ 0 & 2 \end{bmatrix}$.

a) Use Gram-Schmidt process to find an orthonormal basis for the column space of A.

b) Factor A into a product QR, where Q has an orthonormal set of column vectors and R is upper triangular.

Question 8: Let A be an $n \times n$ matrix and λ be an eigenvalue of A.

a) Show that $1/\lambda$ is an eigenvalue of A^{-1}.

b) If $A^2 = A$, show that λ must point either 0 or 1.

Question 9: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix}$.

a) Factor A into a product XDX^{-1}, where D is diagonal.

b) Using XDX^{-1}, find A^6.

Question 10: Let $f(x,y) = \sin x + y^3 + 3xy + 2x - 3y$ and consider the point $P(0,-1)$.

a) Show that P is a stationary point of f.

b) Determine whether P is a local minimum, local maximum, or a saddle point.