KFUPM, DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 232: PROJECT 2, SEMESTER (151),

Name : ....

ID : ....

Exercise 1. Let A, B be two finite sets such that $\mid A \mid= n$ and $\mid B \mid= m$.

(1) Using Mathematical Induction on n, show that the number of functions from A into B is m^n.

(2) Deduce from (1), that the number of subsets of A is 2^n (assign to each subset S of A its characteristic function χ_S)

.

(3) Show that the number of subsets of A of size $p \leq n$ is $\binom{n}{p}$ (use induction on n).

Exercise 2. Let $f : \mathbb{N} \longrightarrow \mathbb{Z}$ be the function defined by $f(n) = \dfrac{(-1)^n(2n-1)+1}{4}$. Show that f is a bijection.

Exercise 3. We define

$$\begin{aligned} g : \mathbb{N} \times \mathbb{N} &\longrightarrow \mathbb{N} \\ (m, n) &\longmapsto 2^{m-1}(2n-1) \end{aligned}$$

Show that g is a bijection.

Exercise 4 (Construction of $\mathbb{Z}$)**.** Let $\mathbb{N} = \{0, 1, 5, 3, \ldots\}$ be the set of all natural numbers. We define the relation R on the cartesian product $\mathbb{N} \times \mathbb{N}$ by:

$$(x, y)R(a, b) \Longleftrightarrow x + b = y + a.$$

(i) Show that R is an equivalence relation.

(ii) If $x, y \in \mathbb{N}$, then we denote by $\overline{(x,y)}$ the equivalence class of (x,y) with respect to the equivalence relation R. Define the addition $\oplus$ on the quotient set $\mathcal{Z} := (\mathbb{N} \times \mathbb{N})/R$ by:

$$\overline{(x,y)} \oplus \overline{(a,b)} = \overline{(x+a, y+b)},$$

and the multiplication $\otimes$ by:

$$\overline{(x,y)} \otimes \overline{(a,b)} = \overline{(xa+yb, xb+ya)}.$$

For $x \in \mathbb{N}$, we denote by $[-x] = \overline{(0,x)}$ and $[x] = \overline{(x,0)}$.

- Explain why $[-x] \oplus [-y] = [-(x+y)]$?
- Explain why $[-x] \otimes [-y] = [xy]$?

(iii) Show that

$$\mathcal{Z} = \{[-x] : x \in \mathbb{N}\} \cup \{[x] : x \in \mathbb{N}\}.$$

(iv) We define the relation $\preceq$ on $\mathcal{Z}$ by:

$\overline{(x,y)} \preceq \overline{(a,b)} \iff$ there esits $k \in \mathbb{N}$ such that $y + a = x + b + k$.

Show that $\preceq$ is a total ordering on $\mathcal{Z}$ and that we have

$$\ldots \preceq [-3] \preceq [-2] \preceq [-1] \preceq [0] \preceq [1] \preceq [2] \preceq [3] \preceq \ldots$$

(v) Show that if $\overline{(c,d)} \in \mathcal{Z}$ and $\overline{(a,b)} \preceq \overline{(x,y)}$, then

$$\overline{(c,d)} \oplus \overline{(a,b)} \preceq \overline{(c,d)} \oplus \overline{(x,y)}.$$

(vi) Show that if $[0] \preceq \overline{(c,d)}$ and $\overline{(a,b)} \preceq \overline{(x,y)}$, then
$$\overline{(c,d)} \otimes \overline{(a,b)} \preceq \overline{(c,d)} \otimes \overline{(x,y)}.$$

(vii) Show that if $\overline{(c,d)} \preceq [0]$ and $\overline{(a,b)} \preceq \overline{(x,y)}$, then
$$\overline{(c,d)} \otimes \overline{(x,y)} \preceq \overline{(c,d)} \otimes \overline{(a,b)}.$$