MATH 202.10 (Term 151)

Quiz 4 (Sects. 4.5 & 4.6) Duration: 20min

Name: ID number:

1.)(5pts)Solve the ODE $y'' - 2y' + y = -8e^{-x} + \cos^2 x$ by using annihilator approach.
2.)(5pts)Solve the ODE $y'' - 4y = \frac{e^{2x}}{e^{2x}+e^{-2x}}$ by using variation of parameters.

1.) First, we solve $y'' - 2y' + y = 0$
Its auxiliary equation is
$m^2 - 2m + 1 = 0$, $m = 1$ (twice)
$\Rightarrow y_c = c_1 e^{x} + c_2 x e^{x}$.

Now, $\cos^2 x = \frac{\cos 2x + 1}{2}$

So, $-8e^{-x} + \cos^2 x = \underbrace{-8e^{-x}}_{D+1} + \underbrace{\frac{1}{2}}_{D} + \underbrace{\frac{1}{2}\cos 2x}_{D^2+4}$

$\Rightarrow D(D+1)(D^2+4)\left[-8e^{-x} + \cos^2 x\right] = 0$

Thus, $\underbrace{(D^2-2D+1)D(D+1)(D^2+4)y = 0}_{\text{New DE}}$

Its auxiliary equation is
$(m-1)^2 m(m+1)(m^2+4) = 0$

$m = 0$, $m = -1$, $m = 1$ (twice), $m = \pm 2i$

$y = \underbrace{c_1 e^{x} + c_2 x e^{x}}_{y_c} + \underbrace{c_3 + c_4 e^{-x} + c_5 \cos 2x + c_6 \sin 2x}_{y_p}$

$y_p = A + Be^{-x} + C\cos 2x + D\sin 2x$

$y_p' = -Be^{-x} - 2C\sin 2x + 2D\cos 2x$

$y_p'' = Be^{-x} - 4C\cos 2x - 4D\sin 2x$

$y_p'' - 2y_p' + y_p = -8e^{-x} + \frac{\cos 2x - 1}{2}$

$A + 4Be^{-x} + (-3C-4D)\cos 2x + (-3D+4C)\sin 2x = -8e^{-x} + \frac{\cos 2x + 1}{2}$

$\Leftrightarrow \begin{cases} A = 1/2 \\ 4B = -8 \Rightarrow B = -2 \\ -3C - 4D = \frac{1}{2} \\ -3D + 4C = 0 \end{cases} \Rightarrow C = -3/50, \quad D = -2/25$

2.) $y'' - 4y = 0$

$\Rightarrow y_c = c_1 \underbrace{e^{2x}}_{y_1} + c_2 \underbrace{e^{-2x}}_{y_2}$.

$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4$

$u_p = u_1 y_1 + u_2 y_2$

$u_1' = \frac{-e^{-2x}}{-4}\left(\frac{e^{2x}}{e^{2x}+e^{-2x}}\right) = \frac{1}{4}\frac{1}{e^{2x}+e^{-2x}} = \frac{1}{4}\frac{e^{2x}}{e^{4x}+1}$

$u_1 = \frac{1}{4}\int \frac{e^{2x}}{e^{4x}+1}\,dx$

$u = e^{2x} \Rightarrow du = 2e^{2x}dx$

$\Rightarrow u_1 = \frac{1}{8}\int \frac{du}{u^2+1} = \frac{1}{8}\tan^{-1} u$

$u_1(x) = \frac{1}{8}\tan^{-1}(e^{2x})$

$u_2' = \frac{e^{2x}}{-4}\frac{e^{2x}}{e^{2x}+e^{-2x}} = -\frac{1}{4}\frac{e^{4x}}{e^{2x}+e^{-2x}} = -\frac{1}{4}\frac{e^{6x}}{e^{4x}+1}$

$\Rightarrow u_2(x) = -\frac{1}{4}\int \frac{e^{6x}}{e^{4x}+1}\,dx$

$u = e^{2x} \rightarrow du = 2e^{2x}dx$

$u_2(x) = -\frac{1}{8}\int \frac{u^2}{u^2+1}\,du = -\frac{1}{8}\int\left(1 - \frac{1}{u^2+1}\right)du$

$= -\frac{1}{8}(u - \tan^{-1} u)$

$= -\frac{1}{8}(e^{2x} - \tan^{-1} e^{2x})$

$\Rightarrow y = y_c + y_p$