Key

Version-FE

King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 201

Final Exam – 2015–2016 (151)

Allowed Time: 180 minutes

Name: ______________________________

ID #: ____________________

Section #: __________ Serial Number: ______________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justification.
3. **Calculators and Mobiles are not allowed.**

Written Problems:

Question #	Grade	Maximum Points
1		12
2		12
3		10
4		10
5		12
6		12
Total:		68

MCQ Problems

Write all your choices in the table below:

Question #	Answer	Grade	Maximum Points
7			06
8			06
9			06
10			06
11			06
12			06
13			06
14			06
15			06
16			06
17			06
18			06
Total:			72

Q:1 (12 points) Find the linearization of $f(x,y,z) = e^z + \cos(x+y)$ at the point $(\frac{\pi}{4}, \frac{\pi}{4}, 0)$.

Sol: $f(\frac{\pi}{4}, \frac{\pi}{4}, 0) = e^0 + \cos\frac{\pi}{2} = 1 + 0 = 1$ (1)

$f_x(x, y, z) = -\sin(x+y)$

$f_x(\frac{\pi}{4}, \frac{\pi}{4}, 0) = -\sin\frac{\pi}{2} = -1$ (02)

$f_y(x, y, z) = -\sin(x+y)$

$f_y(\frac{\pi}{4}, \frac{\pi}{4}, 0) = -1$ (02)

$f_z(x, y, z) = e^z$

$f_z(\frac{\pi}{4}, \frac{\pi}{4}, 0) = 1$ (02)

The linearization of $f(x,y,z)$ at the point $(\frac{\pi}{4}, \frac{\pi}{4}, 0)$ is

(03) $L(x,y,z) = f(\frac{\pi}{4}, \frac{\pi}{4}, 0) + f_x(\frac{\pi}{4}, \frac{\pi}{4}, 0)(x - \frac{\pi}{4})$

$+ f_y(\frac{\pi}{4}, \frac{\pi}{4}, 0)(y - \frac{\pi}{4}) + f_z(\frac{\pi}{4}, \frac{\pi}{4}, 0)(z - 0)$

$= 1 - 1(x - \frac{\pi}{4}) - 1(y - \frac{\pi}{4}) + 1(z - 0)$

(02) $= 1 - x - y + z + \frac{\pi}{2}.$

Q:2 (12 points) Use Lagrange multipliers to find the maximum and minimum values of

$$f(x, y, z) = xz + y^2$$

subject to the constraints $x^2 + y^2 + z^2 = 4$ and $z - x = 0$.

Sol: Let $g(x, y, z) = z - x$

$h(x, y, z) = x^2 + y^2 + z^2 - 4$

We solve the system of equations

$\nabla f = \lambda \nabla g + \mu \nabla h$; $g = 0$, $h = 0$.

Or

$z = -\lambda + 2\mu x$ (i)

$2y = 2\mu y$ (ii) 05

$x = \lambda + 2\mu z$ (iii)

$z - x = 0$ (iv)

$x^2 + y^2 + z^2 = 4$ (v)

(ii) $\Rightarrow$ $y = 0$ or $\mu = 1$. 01

Case I: If $y = 0$, we have $z = x$ (From (iv))

and $2x^2 = 4$ (from (v))

$\Rightarrow x = \pm\sqrt{2}$ 02

$f(\sqrt{2}, 0, \sqrt{2}) = 2$

$f(-\sqrt{2}, 0, -\sqrt{2}) = 2$

Case II: If $\mu = 1$, we have from (i), (iii) and (iv),

$x = 0$, $\lambda = 0$ and $z = 0$. 02

(v) $\Rightarrow$ $y = \pm 2$

$f(0, \pm 2, 0) = 4$.

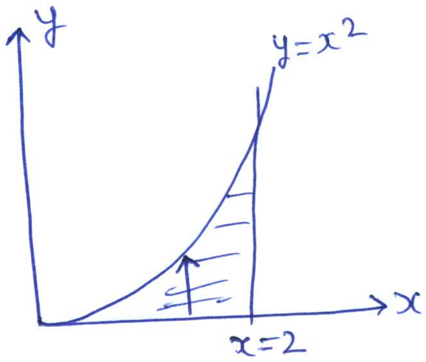
Maximum value $= 4$ 01

Minimum value $= 2$ 01

Q:3 (10 points) Evaluate

$$\int\int_R \cos(x^3)\, dA,$$

where R is the region bounded by $x = 2,\ y = 0$ and $y = x^2$.



$$I = \int\int_R \cos(x^3)\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{x^2} \cos(x^3)\,dy\,dx \qquad 04$$

$$= \int_0^2 y\cos(x^3)\Big|_0^{x^2} dx$$

$$= \int_0^2 x^2\cos(x^3)\,dx \qquad 02$$

$$= \frac{1}{3}\int \cos u\,du$$

$$= \frac{1}{3}\left[\sin(x^3)\right]_0^2 \qquad 02$$

$$= \frac{1}{3}\sin(8) \qquad 02$$

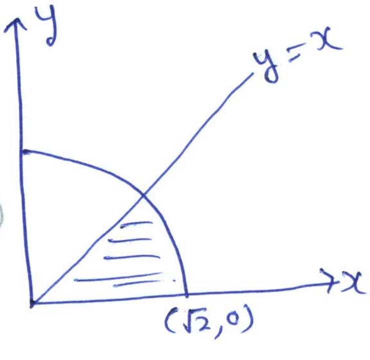
Q:4 (10 points) Evaluate

$$\int\int_R \frac{1}{3+x^2+y^2}\, dA$$

where R is the first quadrant sector of the circle $x^2 + y^2 = 2$ between $y = 0$ and $y = x$.

$$\iint_R \frac{1}{3+x^2+y^2}\, dA$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} \frac{1}{3+r^2} \cdot r\,dr\,d\theta \quad (02)+(02)+(02)$$



$$= \int_0^{\frac{\pi}{4}} \left[\frac{1}{2}\ln(3+r^2)\right]_0^{\sqrt{2}} d\theta \quad (02)$$

$$= \frac{1}{2}(\ln 5 - \ln 3) \int_0^{\pi/4} d\theta \quad (01)$$

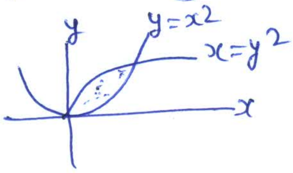
$$= \frac{\pi}{8} \ln\left(\frac{5}{3}\right) \quad (01)$$

Q:5 (12 points) Evaluate

$$\iiint_E x \, dV,$$

where E is bounded by the surfaces $y = x^2$, $x = y^2$, $z = 0$ and $z = x + y$.

$$E = \{(x,y,z) : 0 \le z \le x+y, \; x^2 \le y \le \sqrt{x}, \; 0 \le x \le 1\}$$



$$\iiint_E x \, dV$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} x \, dz \, dy \, dx \qquad 01 + 02 + 01 + 02$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} x(x+y) \, dy \, dx \qquad 02$$

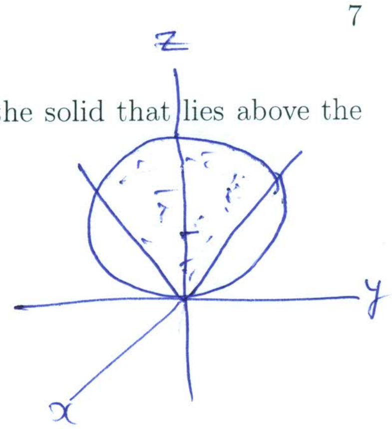
$$= \int_0^1 \left[x^2 y + x \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx \qquad 01$$

$$= \int_0^1 \left[x^{\frac{5}{2}} + \frac{x^2}{2} - x^4 - \frac{x^5}{2} \right] dx \qquad 01$$

$$= \frac{2}{7} x^{\frac{7}{2}} + \frac{x^3}{6} - \frac{x^5}{5} - \frac{x^6}{12} \Big|_0^1 \qquad 01$$

$$= \frac{2}{7} + \frac{1}{6} - \frac{1}{5} - \frac{1}{12} \qquad 01$$

Q:6 (12 points) Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 2z$.



Sol: Sphere $x^2+y^2+(z-1)^2=1$

Sphere in spherical coordinates as

$$\rho^2 = 2\rho\cos\phi \quad \text{or} \quad \rho = 2\cos\phi \qquad 2$$

Equn of cone : $\rho\cos\phi = \rho\sin\phi$

$$\tan\phi = 1$$

$$\phi = \frac{\pi}{4} \qquad 2$$

$$E = \{(\rho, \theta, \phi) : 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/4,\ 0 \le \rho \le 2\cos\phi\}$$

$$\text{Volume} = \iiint_E dV$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \qquad 4$$

$$= 2\pi \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \qquad 1$$

$$= 2\pi \int_0^{\pi/4} \frac{8\cos^3\phi}{3} \cdot \sin\phi \, d\phi \qquad 1 \qquad \text{Let } \cos\phi = u.$$

$$= \frac{16\pi}{3} \left[-\frac{\cos^4\phi}{4} \right]_0^{\pi/4}$$

$$= \frac{16\pi}{12} \left[1 - \frac{1}{4} \right] \qquad 2$$

$$= \frac{16\pi}{12} \cdot \frac{3}{4} = \pi$$

Q:7 (6 points) The length of the curve $x = e^t + e^{-t}$, $y = 5 - 2t$, $0 \le t \le 3$ is

✓ (A) $e^3 - e^{-3}$

(B) $e^2 - e^{-2}$

(C) e^3

(D) 2

(E) e^2

$$L = \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^3 \sqrt{(e^t - e^{-t})^2 + (-2)^2}\, dt$$

$$= \int_0^3 \sqrt{e^{2t} + e^{-2t} - 2 + 4}\, dt$$

$$= \int_0^3 (e^t + e^{-t})\, dt = e^t - e^{-t}\Big|_0^3$$

$$= e^3 - e^{-3} - (1-1)$$

$$= e^3 - e^{-3}$$

Q:8 (6 points) If (a, b, c) is the point of intersection between the line

$$x = 2 - t,\ y = 1 + t,\ z = t, \quad -\infty < t < \infty$$

and the plane $2x - y + 5z = 9$, then $a^2 + b^2 + c^2$ is

✓ (A) 26

(B) 34

(C) 20

(D) 31

(E) 15

$\downarrow$

$$2(2-t) - (1+t) + 5t = 9$$

$$4 - 2t - 1 - t + 5t = 9$$

$$2t = 9 + 1 - 4$$

$$t = 3$$

$$x = -1,\ y = 4,\ z = 3$$

$$(a, b, c) = (-1, 4, 3)$$

$$a^2 + b^2 + c^2 = 1 + 16 + 9 = 26$$

Q:9 (6 points) The area of the region enclosed by one loop of the curve $r = \sin 3\theta$ is

✓(A) $\frac{\pi}{12}$

(B) $\frac{\pi}{4}$

(C) π

(D) $\frac{4\pi}{3}$

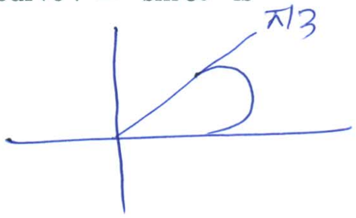
(E) $\frac{\pi}{6}$

$$A = \int_0^{\pi/3} \frac{1}{2}(\sin 3\theta)^2 \, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/3} (1-\cos 6\theta)\, d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3}$$

$$= \frac{1}{4}\left[\frac{\pi}{3} - 0 - 0 - 0\right] = \frac{\pi}{12}$$



Q:10 (6 points) Which different paths would you use to show that $\lim_{(x,y)\to(1,1)} \dfrac{xy^2-1}{y-1}$ does not exist ?

✓(A) $y = \dfrac{1}{x}$ and $y = x$

(B) $y = \dfrac{k}{x}$, k is a constant and $x = 0$

(C) $y = kx$, k is a constant and $y = 0$

(D) $x-$axis and $y-$ axis

(E) $x = y$ and $y = 1$

$$\lim_{(x,y)\to(1,\frac{1}{x})} \frac{xy^2-1}{y-1} = \lim_{x\to 1} \frac{x\frac{1}{x^2}-1}{\frac{1}{x}-1} = 1$$

$$\lim_{(x,y)\to(1,x)} \frac{xy^2-1}{y-1} = \lim_{x\to 1} \frac{x\,x^2-1}{x-1} = \lim_{x\to 1} x^2+1+x$$

$$= 3$$

Q:11 (6 points) A point on the cone $x^2 + y^2 - z^2 = 0$, where the tangent plane is parallel to the plane $3x + 4y + 5z = 0$ is

✓(A) $(3,\ 4,\ -5)$

(B) $(3,\ 0,\ 3)$

(C) $(0,\ 4,\ -4)$

(D) $(5\ ,\ 12,\ 13)$

(E) $(1,\ 1,\ \sqrt{2})$

$\langle 2x, 2y, -2z \rangle = \lambda \langle 3, 4, 5 \rangle$

$x = \frac{3\lambda}{2}$

$y = 2\lambda$

$z = -\frac{5\lambda}{2}$

$\left(\frac{3\lambda}{2}, 2\lambda, -\frac{5\lambda}{2}\right)$ is on the cone.

$\lambda = 2$:

$(3, 4, -5)$

Q:12 (6 points) If $x^3 + z^2 + ye^{xz} + z\cos(y) = 0$, then $\dfrac{\partial z}{\partial y}$ at the point $(0,\ 0,\ 0)$ is

(A) -1

(B) 2

(C) 0

(D) -2

(E) 3

$F = x^3 + z^2 + y e^{xz} + z \cos y$

$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

$= -\frac{(e^{xz} - z \sin y)}{2z + xy e^{xz} + \cos y}$

$\left.\frac{\partial z}{\partial y}\right|_{(0,0,0)} = -\frac{(1-0)}{(0+0+1)} = \frac{-1}{1} = -1$

Q:13 (6 points) The value of the double integral

$$\int_0^8 \int_{x^{\frac{1}{3}}}^2 e^{y^4}\ dydx.$$

is

✓(A) $\dfrac{e^{16}-1}{4}$

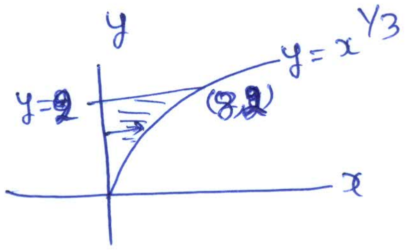
(B) $\dfrac{e^{16}-1}{2}$

(C) $\dfrac{e^{4}-1}{4}$

(D) $e^{16}-1$

(E) $e^{4}-1$

$$\int_{y=0}^{2}\int_{x=0}^{y^3} dx\,(e^{y^4})\,dy$$



$$= \int_0^2 y^3 e^{y^4}\, dy$$

$t = y^4$
$dt = 4y^3 dy$

$$= \int \frac{1}{4} e^t\, dt$$

$$= \frac{1}{4}\left[e^{y^4}\right]_0^2$$

$$= \frac{1}{4}\left[e^{16}-1\right]$$

Q:14(6 points) The area of the region enclosed by $y = x^2+1$, $y = \dfrac{2}{x}$, $x = 0$, $x = 2$ and $y = 0$ is equal to

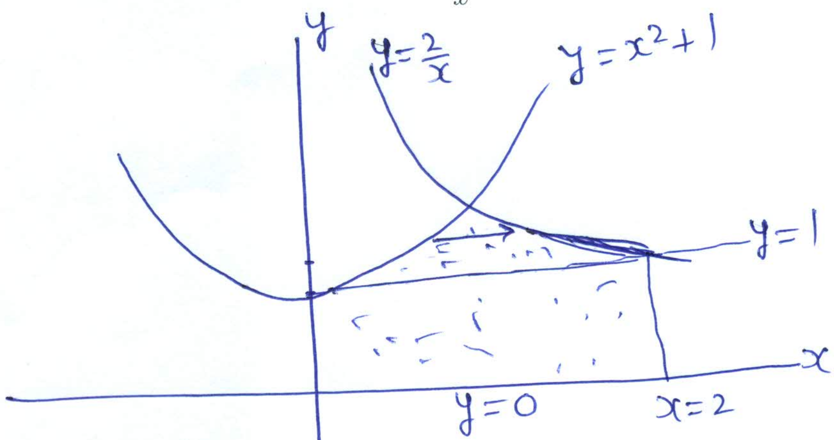
(A) $2 + \int_1^2 \int_{\sqrt{y-1}}^{2/y}\ dxdy$

(B) $2 + \int_1^2 \int_{\sqrt{y-1}}^{2y}\ dxdy$

(C) $2 + \int_1^2 \int_{2/y}^{\sqrt{y-1}}\ dxdy$

(D) $2 + \int_0^2 \int_{\sqrt{y-1}}^{2/y}\ dxdy$

(E) $2 + \int_1^2 \int_{\sqrt{y-1}}^{2/x}\ dxdy$



$$A = \int_{y=1}^{2}\int_{x=\sqrt{y-1}}^{\frac{2}{y}} dx\,dy + 2$$

Q:15(6 points) The local maximum value of the function $f(x, y) = 9 - 2x + 4y - x^2 - 4y^2$ is

(A) 11

(B) 12

(C) 15

(D) 10

(E) 16

$-2 - 2x = 0 \Rightarrow x = -1$

$4 - 8y = 0 \Rightarrow y = \frac{1}{2}$

$f_{xx} = -2 < 0, \; f_{xy} = 0, \; f_{yy} = -8$

$D = f_{xx} f_{yy} - (f_{xy})^2 = 16 > 0$

$f(-1, \frac{1}{2}) = 9 + 2 + 2 - 1 - 1$

$= 13 - 2$

$= 11$

Q:16(6 points) The average value of $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$ on the region between the circles centered at $(0, 0)$ with radii 1 and 2 is

✓(A) $\frac{2}{3}$

(B) $\frac{4}{3}$

(C) $\frac{1}{3}$

(D) 2

(E) 1

$AV = \frac{1}{A} \iint_R f(r, \theta)\, r\, dr\, d\theta$

$= \frac{1}{(4-1)\pi} \int_0^{2\pi} \int_1^2 \frac{1}{r} \cdot r\, dr\, d\theta$

$= \frac{1}{3\pi} \int_0^{2\pi} \int_1^2 dr\, d\theta$

$= \frac{1}{3\pi} \cdot 2\pi\, (2-1) = \frac{2}{3}$

Q:17(6 points) The value of $\int_0^1 \int_0^1 \int_{2y}^2 \frac{3\cos(x^2)}{\sqrt{z}}\, dxdydz$ is

(A) $\frac{3 \sin 4}{2}$

(B) $\frac{3 \sin 4}{5}$

(C) $3 \sin 4$

(D) $\frac{\sin 4}{4}$

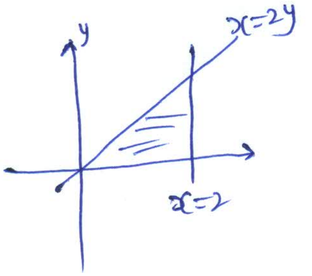
(E) $2 \sin 4$

$$\int_0^1 \int_{x=0}^{2} \int_{y=0}^{\frac{x}{2}} \frac{3\cos(x^2)}{\sqrt{Z}}\, dy\,dx\,dZ$$

$$= 2\int_0^1 \frac{dZ}{2\sqrt{Z}} \int_0^3 \frac{3x\cos(x^2)}{2}\, dx$$

$$= \left(\sqrt{Z}\right)_0^1 (3) \left(\frac{1}{2}\sin(x^2)\right)_0^2$$

$$= \frac{3}{2}\sin 4$$



Q:18(6 points) The integral that gives the volume of the solid bounded by the cylinder $x^2 + y^2 - 2y = 0$ on the lateral sides and bounded on top and bottom by the sphere $x^2 + y^2 + z^2 = 4$ is given by

(A) $\int_0^{\pi} \int_0^{2sin(\theta)} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r dzdrd\theta$

(B) $\int_0^{\pi/2} \int_0^{2sin(\theta)} \int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} r dzdrd\theta$

(C) $\int_0^{\pi/2} \int_0^{2sin(\theta)} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dzdrd\theta$

(D) $\int_0^{\pi} \int_0^{2cos(\theta)} \int_0^{\sqrt{4-r^2}} dzdrd\theta$

(E) $\int_0^{\pi} \int_0^{2cos(\theta)} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r dzdrd\theta$

$x^2 - y^2 - 2y = 0$ or $r^2 - 2r\sin\theta = 0$

$r^2 + Z^2 \le 4$

$r(r - 2\sin\theta) \le 0$

or $Z^2 + r^2 \le 4$

$0 \le r \le 2\sin\theta$

Therefore, $0 \le \theta \le \pi$,

$-\sqrt{4-r^2} \le Z \le \sqrt{4-r^2}$

$$V = \int_0^{\pi} \int_{r=0}^{2\sin\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\, dZ\, dr\, d\theta$$