King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 102 - Exam II - Term 151

Duration: 90 minutes

the KEY

Name:________________ ID Number:________________

Section Number:________________ Serial Number:________________

Class Time:________________ Instructor's Name:________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 7 pages of problems (Total of 7 Problems)

Question Number	Points	Maximum Points
1		16
2		22
3		10
4		10
5		10
6		12
7		20
Total		100

1. Evaluate the following integrals:

(a) **[8 points]** $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx$

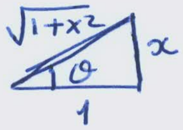
1 Let $x = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

1 $dx = \sec^2\theta \, d\theta$

1 $\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$ as $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

Now $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx = \int \frac{1}{\sec\theta}\sec^2\theta\, d\theta$

$\displaystyle = \int \sec\theta \, d\theta$ 1



$= \ln|\sec\theta + \tan\theta| + C$ 2

$= \ln|\sqrt{1+x^2} + x| + C$ 2

(b) **[8 points]** $\displaystyle\int 3^{x^2+\log_3 x}\,dx$

$\displaystyle = \int 3^{x^2} \cdot 3^{\log_3 x}\, dx$ 2

$\displaystyle = \int 3^{x^2} \cdot x \, dx$ 2

$u = x^2 \Rightarrow du = 2x\,dx$

$\displaystyle = \frac{1}{2}\int 3^u \, du$ 1

$\displaystyle = \frac{1}{2}\frac{3^u}{\ln 3} + C$ 2

$\displaystyle = \frac{1}{2\ln 3} 3^{x^2} + C$ 1

2. (a)[**8 points**] Find a real number a such that $\coth a = \frac{5}{3}$.

$\coth a = \frac{5}{3} \implies \frac{\cosh a}{\sinh a} = \frac{5}{3}$ $\underline{\underline{1}}$

$\implies \frac{e^{a}+e^{-a}}{e^{a}-e^{-a}} = \frac{5}{3}$ $\underline{\underline{2}}$

$\implies \frac{e^{2a}+1}{e^{2a}-1} = \frac{5}{3}$ $\underline{\underline{2}}$ $\implies 3e^{2a}+3 = 5e^{2a}-5$

$\implies e^{2a} = 4$ $\underline{1}$

$\implies 2a = \ln 4$ $\underline{\underline{1}}$

$\implies a = \frac{1}{2}\ln 4$ $\underline{\underline{1}}$

or $\implies a = \ln\sqrt{4} = \ln 2$

(b) [**8 points**] Let $f(x) = \tanh^2 x$. Find $f'(\ln 2)$. (Write your answer as a rational number).

2 $\cdot\ f'(x) = 2\tanh x \cdot \operatorname{sech}^2 x$

$\cdot\ f'(\ln 2) = 2 \cdot \frac{3/4}{5/4} \cdot \frac{1}{\left(\frac{5}{4}\right)^2}$

2 $= \frac{96}{125}$

$\cdot\ \sinh x = \frac{e^{x}-e^{-x}}{2} \implies \sinh(\ln 2) = \frac{2-\frac{1}{2}}{2}$

2 $= \frac{3}{4}$

$\cdot\ \cosh x = \frac{e^{x}+e^{-x}}{2} \implies \cosh(\ln 2) = \frac{2+\frac{1}{2}}{2}$

2 $= \frac{5}{4}$

(c)[**6 points**] Evaluate $\int \frac{\sinh x}{1+\cosh x}\,dx$

Let $u = 1+\cosh x$. Then $du = \sinh x\,dx$. $\underline{\underline{1}}+\underline{\underline{1}}$

$\int \frac{\sinh x}{1+\cosh x}\,dx = \int \frac{1}{u}\,du$ $\underline{\underline{1}}$

$= \ln|u| + C$ $\underline{\underline{2}}$

$= \ln|1+\cosh x| + C$ $\underline{\underline{1}}$

$= \ln(1+\cosh x) + C$, as $1+\cosh x > 0$ for all x

3. [**10 points**] Evaluate $\int 4 \sin^4 t \, dt$.

$$= \int 4 \cdot \left(\sin^2 t\right)^2 \, dt \qquad 1$$

$$= \int 4 \cdot \left(\frac{1-\cos(2t)}{2}\right)^2 \, dt \qquad 2$$

$$= \int 1 - 2\cos(2t) + \cos^2(2t) \quad dt \qquad 2$$

$$= \int 1 - 2\cos(2t) + \frac{1+\cos(4t)}{2} \quad dt \qquad 2$$

$$= \int \frac{3}{2} - 2\cos(2t) + \frac{1}{2}\cos(4t) \quad dt$$

$$= \frac{3}{2}t - \sin(2t) + \frac{1}{8}\sin(4t) + C \qquad 1+1+1$$

4. **[10 points]** Evaluate $\displaystyle\int \frac{x^2+4x+1}{(x-1)(x^2+3)}\,dx$

Use partial fraction decomposition

- $\dfrac{x^2+4x+1}{(x-1)(x^2+3)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+3}$ 1+2

$\Longrightarrow x^2+4x+1 = A(x^2+3) + (Bx+C)(x-1)$

- $x=1 \Longrightarrow 6 = 4A \Longrightarrow A = \frac{3}{2}$ 1
- Coef. of x^2 : $1 = A+B \Longrightarrow B = -\frac{1}{2}$ 1
- Const term : $1 = 3A - C \Longrightarrow C = \frac{9}{2} - 1 \Longrightarrow C = \frac{7}{2}$ 1

- $\displaystyle\int \frac{x^2+4x+1}{(x-1)(x^2+3)}\,dx = \int \frac{3/2}{x-1} + \frac{-\frac{1}{2}x+\frac{7}{2}}{x^2+3}\,dx$

$$= \int \frac{3/2}{x-1} + \frac{-1/2\,x}{x^2+3} + \frac{7/2}{x^2+3}\,dx \quad 1$$

$$= \frac{3}{2}\ln|x-1| - \frac{1}{4}\ln(x^2+3) + \frac{7}{2}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

1 1 1

5. **[10 points]** Evaluate $\int \frac{(x+1)^2\tan^{-1}(3x)+9x^3+x}{(9x^2+1)(x^2+2x+1)}\,dx$

- $9x^3+x = x(9x^2+1)$
- $x^2+2x+1 = (x+1)^2$

$$I = \int \frac{(x+1)^2\tan^{-1}(3x)}{(9x^2+1)(x+1)^2} + \frac{x(9x^2+1)}{(9x^2+1)(x+1)^2}\,dx \quad 2$$

$$= \int \frac{\tan^{-1}(3x)}{9x^2+1} + \frac{x}{(x+1)^2}\,dx \quad 2$$

- $\int \frac{\tan^{-1}(3x)}{9x^2+1}\,dx$; Let $u=\tan^{-1}(3x)$. Then $du = \frac{3}{9x^2+1}\,dx$

3 $= \frac{1}{3}\int u\,du = \frac{1}{6}u^2 + C_1 = \frac{1}{6}\left[\tan^{-1}(3x)\right]^2 + C_1$

- $\int \frac{x}{(x+1)^2}\,dx$; Let $u=x+1$. Then $du=dx$

3 $= \int \frac{u-1}{u^2}\,du = \int \frac{1}{u} - \frac{1}{u^2}\,du = \ln|u| + \frac{1}{u} + C_2$

$= \ln|x+1| + \frac{1}{x+1} + C_2$

$$= \frac{1}{6}\left[\tan^{-1}(3x)\right]^2 + \ln|x+1| + \frac{1}{x+1} + C$$

6. **[12 points]** Evaluate $\int \frac{e^{6x}}{\sqrt{4-e^{4x}}}\,dx$. (Hint: Let $u = e^{2x}$).

Let $u = e^{2x}$.

Then $du = 2e^{2x}\,dx$ 1

$$\int \frac{e^{6x}}{\sqrt{4-e^{4x}}}\,dx = \int \frac{e^{4x}\cdot e^{2x}}{\sqrt{4-e^{4x}}}\,dx$$ 2

$$= \frac{1}{2}\int \frac{u^2}{\sqrt{4-u^2}}\,du$$ 1

Let $u = 2\sin\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ 1

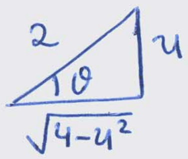
Then, $du = 2\cos\theta\,d\theta$ 1

$\sqrt{4-u^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$ 1

$$= \frac{1}{2}\int \frac{4\sin^2\theta}{2\cos\theta}\cdot 2\cos\theta\,d\theta$$

$$= 2\int \sin^2\theta\,d\theta$$ 1

$$= \int 1-\cos(2\theta)\,d\theta$$ 1



$$= \theta - \frac{1}{2}\sin(2\theta) + C$$ 1

$$= \theta - \sin\theta\cos\theta + C$$ 1

$$= \sin^{-1}\left(\frac{u}{2}\right) - \frac{u}{2}\cdot\frac{\sqrt{4-u^2}}{2} + C$$ 1

$$= \sin^{-1}\left(\frac{e^{2x}}{2}\right) - \frac{1}{4}e^{2x}\sqrt{4-e^{4x}} + C$$

7. Evaluate the improper integral or show that it diverges:

(a) [**8 points**] $\int_0^2 \frac{7}{x-1}\,dx$ $f(x) = \frac{7}{x-1}$ has an infinite disc. at $x=1$.

$$\int_0^2 \frac{7}{x-1}\,dx = \int_0^1 \frac{7}{x-1}\,dx + \int_1^2 \frac{7}{x-1}\,dx \qquad \underline{\underline{2}}$$

$$\int_0^1 \frac{7}{x-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{7}{x-1}\,dx \qquad \underline{\underline{2}}$$

$$= \lim_{t\to 1^-} 7\ln|x-1| \Big]_0^t \qquad \underline{1}$$

$$= \lim_{t\to 1^-} 7\ln|t-1|$$

$$= -\infty$$

Since $\int_0^1 \frac{7}{x-1}\,dx$ diverges, $\underline{1}$

then $\int_0^2 \frac{7}{x-1}\,dx$ diverges. $\underline{1}$

(b)[**12 points**] $\int_1^\infty \ln\left(1+\frac{1}{x^2}\right)\,dx$

$$= \lim_{t\to\infty} \int_1^t \ln\left(1+\frac{1}{x^2}\right)dx \qquad \underline{\underline{2}}$$

by parts: $u = \ln\left(1+\frac{1}{x^2}\right)$, $dv = dx$

$$du = \frac{1}{1+\frac{1}{x^2}} \cdot \frac{-2}{x^3} = \frac{-2}{x^3+x}, \quad v = x$$

$$\int \ln\left(1+\frac{1}{x^2}\right)dx = x\ln\left(1+\frac{1}{x^2}\right) + \int \frac{2x}{x^3+x}\,dx \qquad \underline{\underline{3}}$$

$$= x\ln\left(1+\frac{1}{x^2}\right) + \int \frac{2}{x^2+1}\,dx$$

$$= x\ln\left(1+\frac{1}{x^2}\right) + 2\tan^{-1}x + C \qquad \underline{\underline{2}}$$

$$= \lim_{t\to\infty} \left. x\ln\left(1+\frac{1}{x^2}\right) + 2\tan^{-1}x \right]_1^t$$

$$= \lim_{t\to\infty} t\ln\left(1+\frac{1}{t^2}\right) + 2\tan^{-1}t - \left(\ln 2 + 2\cdot\frac{\pi}{4}\right) \quad \underline{\underline{1}}$$

$$= \underset{\underline{\underline{2}}}{0} + 2\cdot\underset{\underline{\underline{1}}}{\frac{\pi}{2}} - \left(\ln 2 + \frac{\pi}{2}\right)$$

$$= \frac{\pi}{2} - \ln 2 \quad \underline{\underline{1}}$$

So the integral Conv. & Its value is $\frac{\pi}{2} - \ln 2$.

<u>Note</u>:

$$\lim_{t\to\infty} t\ln\left(1+\frac{1}{t^2}\right) \quad , \quad \infty\cdot 0, \text{ indet. form}$$

$$= \lim_{t\to\infty} \frac{\ln\left(1+\frac{1}{t^2}\right)}{\frac{1}{t}} \quad , \quad \frac{0}{0}, \text{ Use L'Hospital's Rule}$$

$$= \lim_{t\to\infty} \frac{\frac{1}{1+\frac{1}{t^2}}\cdot\frac{-2}{t^3}}{-\frac{1}{t^2}}$$

$$= \lim_{t\to\infty} \frac{2}{t^3+t}\cdot t^2 = \lim_{t\to\infty} \frac{2t}{t^2+1} = 0$$