King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 101 Major Quiz

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Question 1[4 points]: Use the Intermediate Value Theorem to show that the functions have an intersection point:

$$f(x) = e^{-x}, \quad \textit{and} \quad g(x) = x^2.$$

Ans: The functions intersect when

$$f(x) = g(x),$$

that is

$$e^{-x} = x^2 \quad \text{or} \quad e^{-x} - x^2 = 0$$

Define $h(x) = e^{-x} - x^2$. We use Intermediate Value Theorem. Note

i) $h(x) = e^{-x} - x^2$ is continuous everywhere

ii) $x = 0$: $h(0) = 1 > 0$

$x = 1$: $h(1) = e^{-1} - 1 < 0$

Then $h(x)$ changes signs on $(0, 1)$.

From the Theorem (IVT), there exists $c \in (0,1)$ such that $h(c) = 0$: $e^{-c} - c^2 = 0$.

Hence, at $x = c$, $f(c) = g(c)$.

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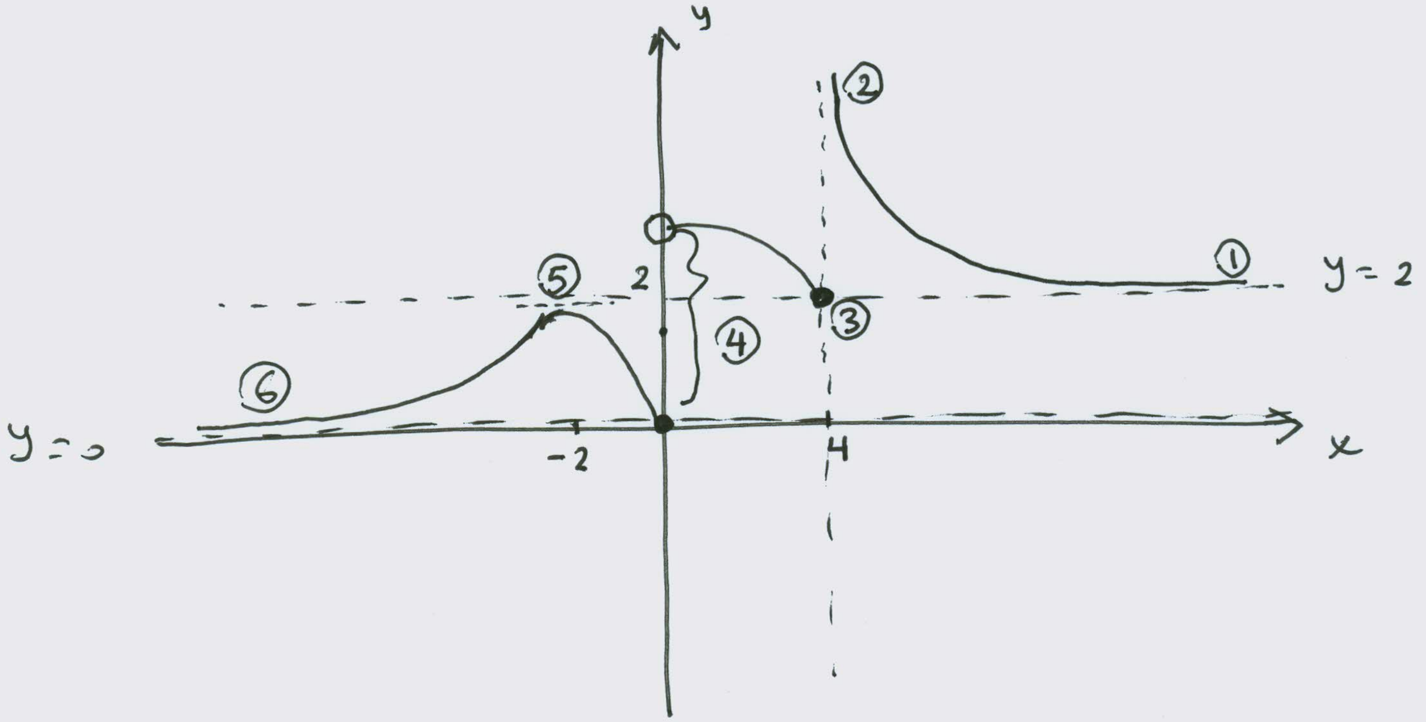
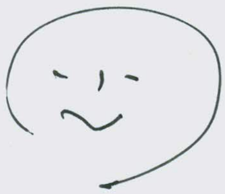
Question 2[6 points]: Graph a function $y = f(x)$ with the following properties:

1. $\lim_{x\to\infty} f(x) = 2$. (HA)
2. $\lim_{x\to 4^+} f(x) = \infty$. (VA)
3. $f(4) = 2$.
4. f has a jump discontinuity at $x = 0$.
5. $f'(-2) = 0$. horizontal tangent at $x = -2$
6. $y = 0$ is a horizontal asymptote. $\Rightarrow \lim_{x\to-\infty} f(x) = 0$

. Start with HAs and VAs





Question 3[4 points]: Find the following limits

1. $\lim_{x\to-\infty} \frac{4x^{2/3}-5x^{1/3}}{6x^{2/3}-7x^{1/3}}$

Ans

$$= \lim_{x\to-\infty} \frac{(4x^{2/3}-5x^{1/3})/x^{2/3}}{(6x^{2/3}-7x^{1/3})/x^{2/3}}$$

$$= \lim_{x\to-\infty} \frac{4-5\underbrace{x^{-1/3}}_{\to 0}}{6-7\underbrace{x^{-1/3}}_{\to 0}} = \frac{4}{6} = \frac{2}{3}.$$

<u>Remark</u> : $y = \frac{2}{3}$ is a HA for $f(x) = \dfrac{4x^{2/3}-5x^{1/3}}{6x^{2/3}-7x^{1/3}}$

2. $\lim_{x\to 0}(\frac{1}{x} - \frac{\cos x}{x})$

<u>Ans</u>:

$$= \lim_{x\to 0} \frac{1-\cos x}{x} \qquad \frac{0}{0}\,!$$

$$= \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}$$

Note $\sin^2 x \neq \sin x^2$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \underbrace{\sin x}_{\to 0} \cdot \underbrace{\frac{\sin x}{x}}_{\to 1} \underbrace{\frac{1}{1+\cos x}}_{\to \frac{1}{2}}$$

$$= 0\,(1)\left(\tfrac{1}{2}\right) = 0.$$

Question 4[6 points]: Determine all asymptotes of the function

$$f(x) = \frac{|x^3 - 1|}{x^3 - x}$$

Ans: • Horizontal Asymptotes (HA)

$$\lim_{x \to \pm\infty} \frac{|x^3 - 1|}{x^3 - x} = \lim_{x \to \pm\infty} \frac{\pm(x^3 - 1)}{x^3 - x}$$ Divide by x^3

$$= \lim_{x \to \pm\infty} \frac{\pm(1 - \overbrace{1/x^3}^{\to 0})}{1 - \underbrace{1/x^2}_{\to 0}} = \pm 1 \Rightarrow y = 1,\ y = -1$$ are the horizontal asymptotes.

• Vertical Asymptotes (VA) $f(x) = \dfrac{|(x-1)||x^2+x+1|}{x(x-1)(x+1)}$.

Test $x = 0$, $x = 1$, $x = -1$

i) $$\lim_{x \to 0^\pm} \frac{|x-1||x^2+x+1|}{x(x-1)(x+1)} = \frac{(1)(1)}{0^\pm(-1)(1)} = \mp\infty$$

$\Rightarrow$ $x = 0$ is a VA.

ii) $$\lim_{x \to 1^\pm} \frac{|x-1||x^2+x+1|}{x(x-1)(x+1)} = \lim_{x \to 1^\pm} \pm\frac{\cancel{(x-1)}|x^2+x+1|}{x\cancel{(x-1)}(x+1)}$$

$= \pm\frac{3}{2} \neq \pm\infty \Rightarrow x = 1$ is not a VA.

iii) $$\lim_{x \to -1^\pm} \frac{|x-1||x^2+x+1|}{x(x-1)(x+1)} = \frac{2(1)}{(-1)(-2)0^\pm} = \pm\infty$$

$\Rightarrow$ $x = -1$ is a VA.

Good Luck

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Math 101 Quiz 2

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Question 1: Let $p(x)$ be a positive polynomial, and

$$\lim_{x\longrightarrow a} \frac{4+\sqrt{p(x)}}{1+\sqrt{p(x)}} = 2.$$

Find $p(a)$.

Ans: $\lim_{x \to a} \frac{4+\sqrt{p(x)}}{1+\sqrt{p(x)}} = 2$

$\Rightarrow \frac{4+\sqrt{p(a)}}{1+\sqrt{p(a)}} = 2$

$4+\sqrt{p(a)} = 2+2\sqrt{p(a)}$

$\Rightarrow \sqrt{p(a)} = 2$

$\Rightarrow \boxed{p(a) = 4}$

Question 2: Use the Sqeeze Theorem to find

$$\lim_{x\longrightarrow 0} \left[\cos(\frac{1}{x})\cos(\frac{\pi}{2}+x)\right].$$

Ans: Note $-\left|\cos(\frac{\pi}{2}+x)\right| \le \cos(\frac{1}{x})\cos(\frac{\pi}{2}+x) \le \left|\cos(\frac{\pi}{2}+x)\right|$,

and $\lim_{x \to 0} -\left|\cos(\frac{\pi}{2}+x)\right| = 0$, $\lim_{x \to 0} \left|\cos(\frac{\pi}{2}+x)\right| = 0$.

Hence, by the Theorem; $\lim_{x \to 0} \cos(\frac{1}{x})\cos(\frac{\pi}{2}+x) = 0$

Good Luck