King Fahd University Of Petroleum & Minerals
Department Of Mathematics And Statistics

MATH 101-Exam I -Term (151)
Duration: 90 minutes

KEY- SOLUTION

Name: ______________________ ID: ______________________

Section Number: ______________________ Serial Number: ______________________

Class Time: ______________________ Instructor's Name: ______________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure you have 7 pages of problems (Total of 7 Problems)

Question Number	Full Mark	Marks Obtained
One	29	
Two	10	
Three	18	
Four	8	
Five	12	
Six	11	
Seven	12	
Total	100	

Question.1 (5+7+8+9=29-Points)
Evaluate the following limits, if exist .

(a) $\lim\limits_{x\to2}\dfrac{x^3+x^2-6x}{2-x}$ $\rightarrow \frac{0}{0}$

$$\text{Lim}\ \frac{x^3+x^2-6x}{2-x} = \lim_{x\to2}\frac{x(x^2+x-6)}{2-x} \quad \} \text{ 1 pt}$$

$$= \lim_{x\to2}\frac{x(x-2)(x+3)}{2-x} \quad \} \text{ 1 pt}$$

$$= \lim_{x\to2} -x(x+3) \quad \} \text{ 2 pts}$$

$$= -2(2+3) = -10 \quad \} \text{ 1 pt}$$

(b) $\lim\limits_{z\to2+}\dfrac{x-2}{|4-x^2|}$ $\rightarrow \frac{0}{0}$

$$|4-x^2| = \begin{cases} 4-x^2, & 4-x^2>0 \\ -(4-x^2), & 4-x^2\le 0 \end{cases} = \begin{cases} 4-x^2, & 4>x^2 \\ x^2-4, & 4\le x^2 \end{cases}$$

$$= \begin{cases} 4-x^2, & -2<x<2 \\ x^2-4, & x\geq 2,\ x\le -2 \end{cases}$$

$$\therefore \lim_{x\to2^+}\frac{x-2}{|4-x^2|} = \lim_{x\to2^+}\frac{x-2}{x^2-4} \quad \} \text{ 4 pts}$$

$$= \lim_{x\to2^+}\frac{x-2}{(x-2)(x+2)} \quad \} \text{ 1 pt}$$

$$= \lim_{x\to2^+}\frac{1}{x+2} \quad \} \text{ 1 pt}$$

$$= \frac{1}{2+2} = \frac{1}{4}. \quad \} \text{ 1 pt}$$

(c) $\lim_{x\to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}} \to \frac{0}{0}$

$$\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}} = \lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}}\cdot\frac{\sqrt{x}+x^2}{\sqrt{x}+x^2} \Big\} \text{ 2 pts}$$

$$= \lim_{x\to 1}\frac{(x-x^4)(1+\sqrt{x})}{(1-x)(\sqrt{x}+x^2)} \Big\} \text{ 1 pt}$$

$$= \lim_{x\to 1}\frac{1+\sqrt{x}}{\sqrt{x}+x^2}\cdot\frac{x(1-x^3)}{1-x} \Big\} \text{ 1 pt}$$

$$= \lim_{x\to 1}\frac{1+\sqrt{x}}{\sqrt{x}+x^2}\cdot\frac{x(1-x)(1+x+x^2)}{(1-x)} \Big\} \text{ 1 pt}$$

$$= \lim_{x\to 1}\frac{1+\sqrt{x}}{\sqrt{x}+x^2}\cdot x(1+x+x^2) \Big\} \text{ 1 pt}$$

$$= \frac{1+1}{1+1}\cdot 1(1+1+1) \Big\} \text{ 1 pt}$$

$$= 3 \Big\} \text{ 1 pt}$$

(d) $\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right) \to (\infty-\infty)$

$$\lim_{x\to\infty}(\sqrt{9x^2+x}-3x) = \lim_{x\to\infty}(\sqrt{9x^2+x}-3x)\frac{(\sqrt{9x^2+x}+3x)}{\sqrt{9x^2+x}+3x} \Big\} \text{ 2 pts}$$

$$= \lim_{x\to\infty}\frac{9x^2+x-9x^2}{\sqrt{9x^2+x}+3x} \Big\} \text{ 1 pt}$$

$$= \lim_{x\to\infty}\frac{x}{|x|\sqrt{9+\frac{1}{x}}+3x} \Big\} \text{ 1 pt}$$

AS $x\to\infty$, $|x| = x$ $\Big\}$ 1 pt

$$= \lim_{x\to\infty}\frac{x}{x(\sqrt{9+\frac{1}{x}}+3)} \Big\} \text{ 2 pts}$$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{9+\frac{1}{x}}+3} \Big\} \text{ 1 pt}$$

$$= \frac{1}{3+3} = \frac{1}{6} \Big\} \text{ 1 pt}$$

Question.2 **(10-Points)**
Let

$$f(x) = \begin{cases} a + x & , x < -1 \\ x^2 & , -1 \leq x < 1 \\ b - x & , x \geq 1 \end{cases}$$

Find all values of a and b such that $\lim_{x\to-1} f(x)$ **exists** and $\lim_{x\to 1} f(x)$ **does not exist.**

$\lim_{x\to-1}$ exists $\Rightarrow \lim_{x\to-1^-} f(x) = \lim_{x\to-1^+} f(x)$ } 1 pt

$\lim_{x\to-1^-} f(x) = \lim_{x\to-1} (a+x) = a-1$ } 1 pt

$\lim_{x\to-1^+} f(x) = \lim_{x\to-1} x^2 = (-1)^2 = 1$ } 1 pt

$\therefore \quad a-1 = 1$ } 1 pt

$\Rightarrow \boxed{a=2}$ } 1 pt

To have $\lim_{x\to1} f(x)$ does not exist, $\lim_{x\to1^-} f(x) \neq \lim_{x\to1^+} f(x)$ } 1 pt

$\lim_{x\to1^-} f(x) = \lim_{x\to1} x^2 = (1)^2 = 1$ } 1 pt

$\lim_{x\to1^+} f(x) = \lim_{x\to1} (b-x) = b-1$ } 1 pt

$\therefore \quad b-1 \neq 1$ } 1 pt

$\Rightarrow \boxed{b \neq 2}$ } 1 pt

(or $b = (-\infty, 2) \cup (2, \infty)$)

Question 3. **(10+8=18-Points)**

(a) Use the graph of $y = f(x) = \dfrac{2}{\sqrt{x}}$, $x > 0$ to find a number $\delta > 0$ such that:

If $0 <| x - 1 |< \delta$ then $| f(x) - 2 |< \dfrac{1}{2}$.

$f(x_1) = \dfrac{2}{\sqrt{x_1}} = 2.5 = \dfrac{25}{10} = \dfrac{5}{2}$ } 1 pt

$\sqrt{x_1} = \dfrac{4}{5} \Rightarrow x_1 = \dfrac{16}{25}$ } 1 pt

$f(x_2) = \dfrac{2}{\sqrt{x_2}} = 1.5 = \dfrac{15}{10} = \dfrac{3}{2}$ } 1 pt

$\sqrt{x_2} = \dfrac{4}{3} \Rightarrow x_2 = \dfrac{16}{9}$ } 1 pt

$\delta_1 = 1 - x_1$ } 1 pt

$= 1 - \dfrac{16}{25} = \dfrac{9}{25}$ } 1 pt

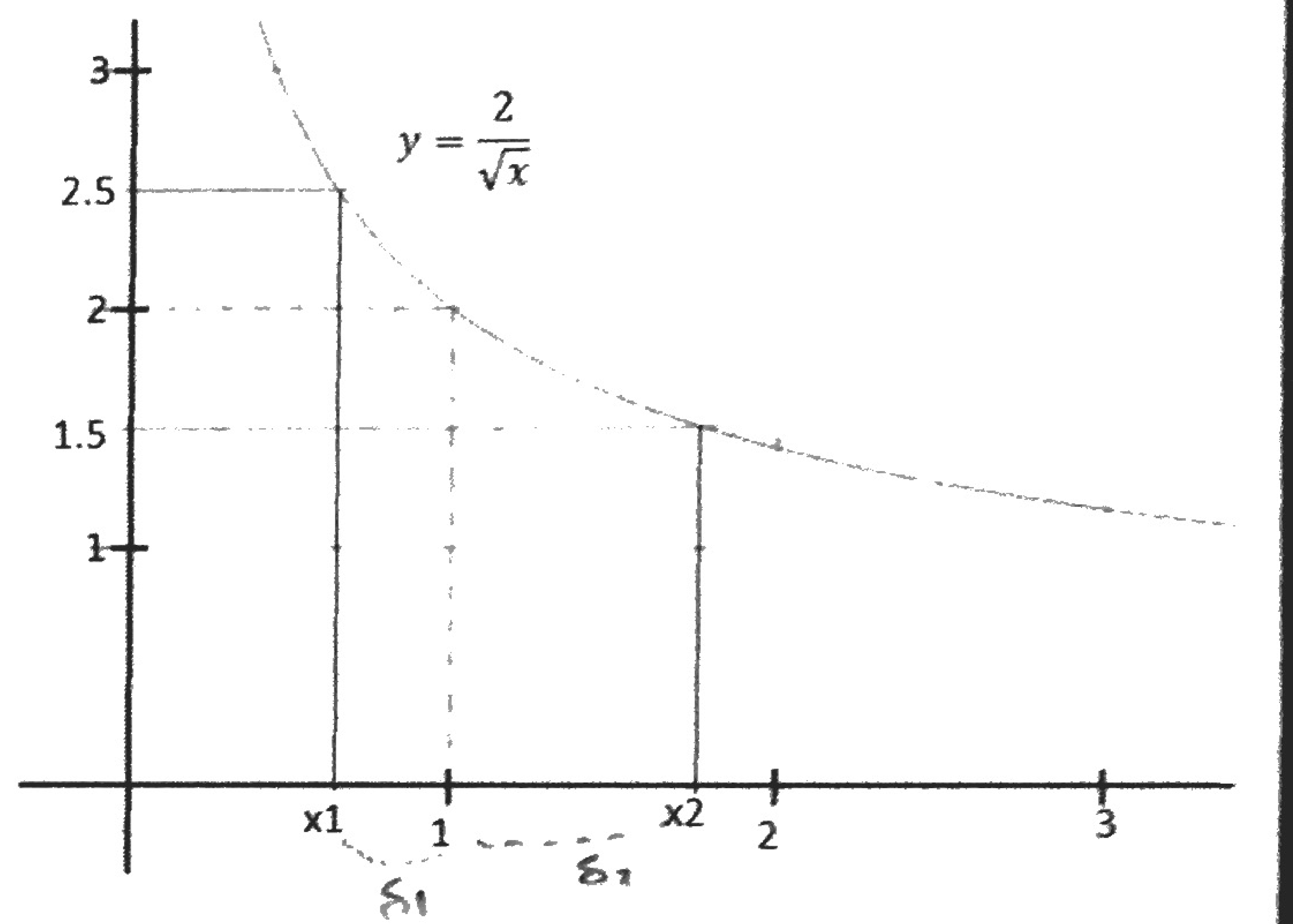
$\delta_2 = x_2 - 1$ } 1 pt

$= \dfrac{16}{9} - 1 = \dfrac{7}{9}$ } 1 pt

$\delta = \min.\{\delta_1, \delta_2\}$

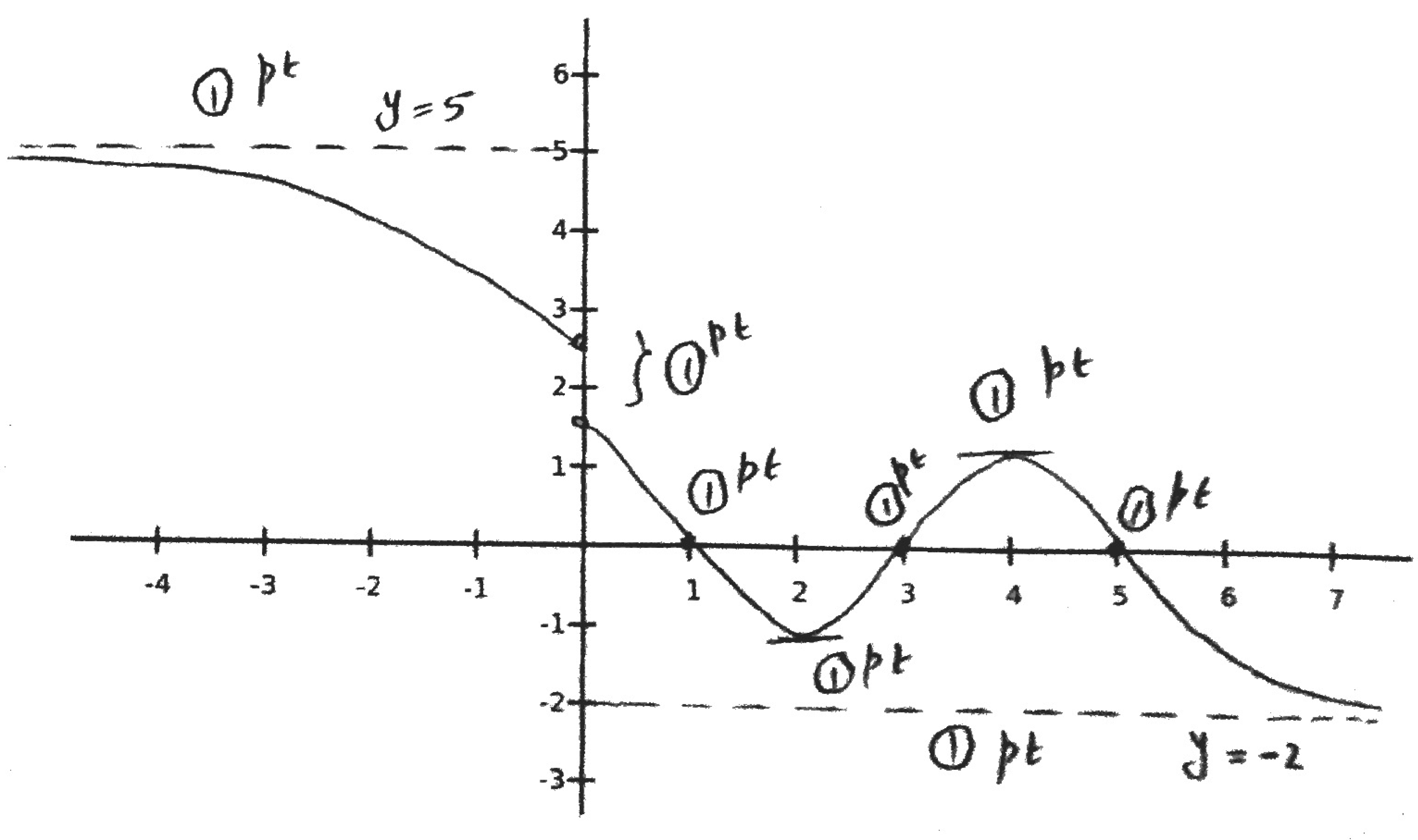
$= \min\{\dfrac{9}{25}, \dfrac{7}{9}\}$ } 1 pt

$= \dfrac{9}{25}$ } 1 pt (or any smaller positive number)



(b) Sketch the graph of a[redacted] function $g(x)$ for which:
$g(1) = g(3) = g(5) = 0$,
$g'(2) = g'(4) = 0$,
$g(x)$ has jump discontinuity at $x = 0$,
$\lim\limits_{x\to\infty} g(x) = -2$, $\lim\limits_{x\to-\infty} g(x) = 5$.

(One-possible graph)



Question 4. **(8-Points)**
Use the Intermediate Value Theorem to show that there is a root of the equation:
$\sin(x) = 4 - x - 3\sqrt{x}$ in the interval $(0, 1)$.

Write the equation as: $\sin x - 4 + x + 3\sqrt{x} = 0$

Let $f(x) = \sin x - 4 + x + 3\sqrt{x}$, $x \in [0,1]$ } 2 pts

(1) $f(x)$ is conts. on $[0,1]$ / since $\sin x$, $-4+x$, $3\sqrt{x}$ are all conts. on $[0,1]$ } 1 pt

(2) $f(0) = 0 - 4 + 0 + 0 = -4$ } 1 pt
$f(1) = \sin 1 - 4 + 1 + 3 = \sin 1$ } 1 pt

(3) $N = 0$ lies between $f(0)$ and $f(1)$ } 1 pt

$\therefore$ $f(x)$ satisfies the conditions of the IVT
$\Rightarrow$ There is at least $c \in (0,1)$ such that
$f(c) = 0$
$\sin c = 4 - c - 3\sqrt{c}$ } 2 pts
or the equation has a root in the interval $(0,1)$.

Question 5. **(12-Points)**
Use limits to find all horizontal and vertical asymptotes to the curve of the function:
$$f(x) = \frac{x^2 - 3x + 2}{2x^2 - 3x + 1}$$

$f(x) = \dfrac{(x-1)(x-2)}{(2x-1)(x-1)}$ } 2 pt

$= \dfrac{x-2}{2x-1}$, $x \neq 1, \frac{1}{2}$ } 2 pts

$\boxed{1}$ V.A's: (i) when $x = 1$

$\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \dfrac{x-2}{2x-1} = \dfrac{-1}{1} = -1$ } 1 pt

$\therefore$ NO V.A when $x = 1$ } 1 pt

(ii) when $x = \frac{1}{2}$

$\lim\limits_{x \to \frac{1}{2}^+} f(x) = \lim\limits_{x \to \frac{1}{2}^+} \dfrac{(x-2)}{2x-1} = \dfrac{-3/2}{0^+} = -\infty$ } 1 pt

$\therefore$ $x = \frac{1}{2}$ is a V.A. } 1 pt

$\boxed{2}$ H.A's: (i) $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \dfrac{x-2}{2x-1} = \lim\limits_{x \to \infty} \dfrac{1 - 2/x}{2 - \frac{1}{x}} = \frac{1}{2}$ } 1 pt

$\therefore$ $y = \frac{1}{2}$ is a H.A. } 1 pt

(ii) $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \dfrac{x-2}{2x-1} = \frac{1}{2}$ } 1 pt

$\therefore$ $y = \frac{1}{2}$ is the only H.A. } 1 pt

Question 6. **(6+5=11-Points)**
Let

$$f(x) = \begin{cases} \frac{(x-1)(x+3)}{(x-1)^n} & , x > 1 \\ x^2 + 3 & , x \leq 1 \end{cases}$$

where n is a nonnegative integer, $(n \geq 0)$.

(a) Use limits to find the value(s) of n for which the function is **continuous at every** x.

(1) $\frac{(x-1)(x+3)}{(x-1)^n}$. is conts on its domain $(1,\infty)$ } 1 pt

(2) x^2+3 : is conts on its domain $(-\infty,1]$ } 1 pt

(3) when $x=1$
We must have $\lim_{x\to1} f(x) = f(1) = 4$ } 1 pt

$\Rightarrow \lim_{x\to1^+} f(x) = \lim_{x\to1^-} f(x)$

$\lim_{x\to1^-} f(x) = \lim_{x\to1} (x^2+3) = 1+3 = 4$ } 1 pt

$\lim_{x\to1^+} f(x) = \lim_{x\to1} \frac{(x-1)(x+3)}{(x-1)^n}$

$= \lim_{x\to1^+} (x-1)^{1-n}(x+3) = \lim_{x\to1^-} f(x) = 4$ } 1 pt

if and only if $1-n=0 \Leftrightarrow n=1$ } 1 pt

(b) Use limits to find the value(s) of n for which the function has **infinite discontinuity at** $x = 1$.

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \frac{(x-1)(x+3)}{(x-1)^n}$ } 1 pt

1 pt

$= \lim_{x\to1^+} \frac{(x+3)}{(x-1)^{n-1}} = +\infty$ if and only if

$n-1 \geq 1$ } 2 pts

$\Rightarrow n \geq 2$ } 1 pt

Question 7. **(8+4=12-Points)**

Let $f(x) = \dfrac{x}{2x+1}$

(a) Use the definition of the derivative to find $f'(1)$.

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \quad \} \text{ 1 pt}$$

$$= \lim_{h\to 0} \frac{\frac{1+h}{2+2h+1} - \frac{1}{2+1}}{h} \quad \} \text{ 1 pt}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1+h}{3+2h} - \frac{1}{3}\right] \quad \} \text{ 1 pt}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{3+3h-3-2h}{3(3+2h)}\right] \quad \} \text{ 2 pts}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h}{3(3+2h)}\right] \quad \} \text{ 1 pt}$$

$$= \lim_{h\to 0} \frac{1}{3(3+2h)} \quad \} \text{ 1 pt}$$

$$= \frac{1}{3(3)} = \frac{1}{9}. \quad \} \text{ 1 pt}$$

(b) Use part (a) to find an equation of the tangent line to the curve $y = f(x)$ at the point $(1, \frac{1}{3})$.

The tangent equation is:

$$y - y_1 = m(x - x_1)$$

part (a) $\rightarrow$ $m = f'(1) = \frac{1}{9}$ $\}$ 2 pts

$$\therefore \; y - \frac{1}{3} = \frac{1}{9}(x-1) \quad \} \text{ 1 pt}$$

$$y = \frac{1}{9}x - \frac{1}{9} + \frac{1}{3}$$

$$y = \frac{1}{9}x + \frac{2}{9} \quad \} \text{ 1 pt}$$