King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 202 - Final Exam - Term 143

Duration: 180 minutes

Name: Key

ID Number:

Section Number:

Serial Number:

Class Time:

Instructor's Name:

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 12 pages of problems (Total of 12 Problems)

Question Number	Points	Maximum Points
1		12
2		12
3		10
4		14
5		14
6		10
7		12
8		12
9		14
10		10
11		10
12		10
Total		150

1. **(12 points)** Find the general solution of the homogeneous linear system $X' = AX$ where $A = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$.

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0 \qquad (2\text{ pts})$$

$$\Rightarrow (5-\lambda)(3-\lambda)+2=0$$

$$\Rightarrow \lambda^2 - 8\lambda + 17 = 0 \qquad (1\text{ pts})$$

$$\Rightarrow \lambda = \frac{8 \pm \sqrt{64-68}}{2} = 4 \pm i \qquad (1\text{ pt})$$

The eigenvector corresponds to $\lambda = 4+i$:

$$\left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right] \Rightarrow (1-i)K_1 = -K_2 \qquad (2\text{ pts})$$

$$\Rightarrow K_1 = \frac{-1}{1-i}K_2 \quad . \text{ let } K_2 = 1-i$$

$$\Rightarrow K_1 = -1$$

$$\therefore K_{\lambda=4+i} = \begin{pmatrix} -1 \\ 1-i \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} i \qquad (2\text{pts})$$

$$\Rightarrow X_1 = e^{4t}\left[\begin{pmatrix} -1 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin t\right] \qquad (2\text{ pts})$$

and

$$X_2 = e^{4t}\left[\begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos t + \begin{pmatrix} -1 \\ 1 \end{pmatrix}\sin t\right] \qquad (2\text{ pts})$$

$$X = c_1 X_1 + c_2 X_2$$

2. **(12 points)** Use variation of parameters to find a particular solution X_p of the nonhomogeneous system $X' = AX + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$ where

$X_c = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$ form a general solution of the associated homogeneous system.

$$\phi(t) = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix} \quad (2\text{ pts})$$

$$\phi^{-1}(t) = \frac{1}{e^{3t}} \begin{pmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{pmatrix} = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix} \quad (2\text{pts})$$

$$X_p = \phi(t) \int \phi^{-1}(t) F(t)\, dt \quad (2\text{ pts})$$

$$\phi^{-1}(t) F(t) = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix} \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -3e^{-t} \end{pmatrix} \quad (2\text{pts})$$

$$\Rightarrow \int \phi^{-1}(t) F(t)\, dt = \begin{pmatrix} 2t \\ 3e^{-t} \end{pmatrix} \quad (2\text{pts})$$

$$\Rightarrow X_p = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix} \begin{pmatrix} 2t \\ 3e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} 4te^t + 3e^t \\ 2te^t + 3e^t \end{pmatrix} \quad (2\text{pts})$$

$$= \begin{pmatrix} 4 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^t$$

3. **(10 points)** Use matrix exponential to solve the initial-value problem
$X' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} X$ subject to $X(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 2^2 & 0 \\ 0 & (-3)^2 \end{pmatrix},$$

$$A^n = \begin{pmatrix} 2^n & 0 \\ 0 & (-3)^n \end{pmatrix}. \quad (2\text{ pts})$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$$

$$= \begin{pmatrix} 1 + 2t + \frac{(2t)^2}{2!} + \cdots + \frac{(2t)^n}{n!} + \cdots & 0 \\ 0 & 1 + (-3t) + \cdots + \frac{(-3t)^n}{n!} + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \quad (3\text{ pts})$$

$$X = e^{At} \cdot C = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \quad (2\text{ pts})$$

$$= \begin{pmatrix} C_1 e^{2t} \\ C_2 e^{-3t} \end{pmatrix} = C_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ e^{-3t} \end{pmatrix}$$

$$X(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow$$

$$C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow C_1 = 2, \quad (1\text{ pt})$$
$$C_2 = 3 \quad (1\text{ pt})$$

$$\therefore \; X = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{2t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} e^{-3t} \quad (1\text{ pt})$$

4. **(14 points)** Find the general solution of the homogeneous linear system $X' = \begin{pmatrix} -6 & 10 \\ -2 & 6 \end{pmatrix} X$.

$$|A-\lambda I| = \begin{vmatrix} -6-\lambda & 10 \\ -2 & 6-\lambda \end{vmatrix} = \lambda^2 - 36 + 20 = 0 \quad (2pts)$$

$$\Rightarrow \lambda^2 = 16 \Rightarrow \lambda = \pm 4. \quad (2pts)$$

For $\lambda = 4$:

$$\left(\begin{array}{cc|c} -10 & 10 & 0 \\ -2 & 2 & 0 \end{array}\right) \Rightarrow -K_1 + K_2 = 0 \quad (2pts)$$

$$\Rightarrow K_1 = K_2$$

$$\therefore K_{\lambda=4} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (2pts)$$

For $\lambda = -4$

$$\left(\begin{array}{cc|c} -2 & 10 & 0 \\ -2 & 10 & 0 \end{array}\right) \Rightarrow -2K_1 + 10K_2 = 0 \quad (2pts)$$

$$\Rightarrow K_1 = 5K_2$$

$$K_{\lambda=4} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad (2pts)$$

So, the general solution is

$$X = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 5 \\ 1 \end{pmatrix} e^{-4t} \quad (2pts)$$

5. **(14 points)** Find the first four nonzero terms of the series solution of the equation $3xy'' + (2-x)\,y' - y = 0$ which corresponds to the indical root $r = \dfrac{1}{3}$ of the singularity $x = 0$.

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r} \Rightarrow y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$,

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} \quad (2\text{pts})$$

substituting in the D.E, to get

$$\sum_{n=0}^{\infty} 3a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n (n+r) x^{n+r-1} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r}$$

$$- \sum_{n=0}^{\infty} a_n x^{n+r} = 0 . \quad (2\text{pts})$$

$$\Rightarrow x^r \left[\sum_{n=0}^{\infty} a_n (n+r)(3n+3r-1) x^{n-1} + \sum_{n=0}^{\infty} a_n (n+r+1) x^n \right] = 0$$

(2pts)

$$\Rightarrow r(3r-1) a_0 x^{-1} + \sum_{n=1}^{\infty} a_n (n+r)(3n+3r-1) x^{n-1} - \sum_{n=0}^{\infty} a_n (n+r+1) x^n = 0$$

$$\Rightarrow r(3r-1) a_0 x^{-1} + \sum_{n=0}^{\infty} a_{n+1} (n+r+1)(3n+3r+2) x^n - \sum_{n=0}^{\infty} a_n (n+r+1) x^n = 0$$

(1pt) (1pt) (2pts)

$$\Rightarrow r = 0 \text{ or } \boxed{r = \tfrac{1}{3}} , \text{ and } a_{n+1} = \frac{1}{3n+3r+2} a_n , \; n = 0,1,2,\ldots$$

For $r = \frac{1}{3} \Rightarrow a_{n+1} = \dfrac{1}{3n+3} a_n , \; n = 0, 1, 2, \ldots$ (2pts)

$$a_1 = \frac{1}{3} a_0 , \quad a_2 = \frac{1}{6} a_1 = \frac{1}{18} a_0 , \quad a_3 = \frac{1}{162} a_0$$

$$\therefore \; y = a_0 x^{\frac{1}{3}} \left[1 + \frac{1}{3} x + \frac{1}{18} x^2 + \frac{1}{162} x^3 + \ldots \right] \quad (2\text{pts})$$

6. a) **(5 points)** Find a homogeneous Cauchy-Euler differential equation whose solution is $y = c_1x^3 + c_2x^5$.

Let the D.E be $ax^2y'' + bxy' + cy = 0$ (1pt)

So, the auxiliary equation is

$am^2 + (b-a)m + c = 0$

$m_1 = 3,\ m_2 = 5 \Rightarrow (m-3)(m-5) = 0$ (1pt)

$\Rightarrow m^2 - 8m + 15 = 0$, so we may take

$a = 1$, $b - a = -8 \Rightarrow b = -7$ and $c = 15$

(1pt) (1pt) (1pt)

So, a C.E. equation is $x^2y'' - 7xy' + 15y = 0$

b) **(5 points)** Determine whether the set of functions $\{e^x, e^{-x}, \cosh x\}$ is linearly independent on the interval $(-\infty, \infty)$.

the set is Linearly dependent since (2pts)

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$= \frac{1}{2}(e^x) + \frac{1}{2}(e^{-x}) \quad (3\text{ pts})$$

So, $y_1(x) = \cosh x$ is a Linear Combination

of $y_2(x) = e^x$ and $y_3(x) = e^{-x}$.

7. **(12 points)** Find the general solution of $y'' + 2y' + y = e^{-x} \ln x$.

$y'' + 2y' + y = 0$

$m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0$ (1 pts)

$\Rightarrow m = -1, -1$

$\Rightarrow y_c = c_1 e^{-x} + c_2 x e^{-x}$ (1 pts)

$\therefore y_1(x) = e^{-x}, \quad y_2(x) = x e^{-x}$

let $y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$

$$W = \begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{vmatrix} = e^{-2x} \quad (2 \text{ pts})$$

$$W_1 = \begin{vmatrix} 0 & x e^{-x} \\ e^{-x} \ln x & e^{-x} - x e^{-x} \end{vmatrix} = -x e^{-2x} \ln x \quad (2 \text{pts})$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x} \ln x \end{vmatrix} = e^{-2x} \ln x \quad (1 \text{ pt})$$

$$u_1(x) = \int \frac{W_1}{W} dx = \int -x \ln x \, dx = -\frac{x^2}{2} \ln x + \frac{x^2}{4} \quad (2 \text{pts})$$

$$u_2(x) = \int \frac{W_2}{W} dx = \int \ln x \, dx = x \ln x - x \quad (2 \text{pts})$$

$$\therefore y = c_1 e^{-x} + c_2 x e^{-x} + \left(-\frac{x^2}{2} \ln x + \frac{x^2}{4}\right) e^{-x} + (x^2 \ln x - x^2) e^{-x}$$

$$= c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{2} x^2 e^{-x} \ln x - \frac{3}{4} x^2 e^{-x} \quad (1 \text{ pt})$$

8. **(12 points)** Find the general solution of the differential equation
$(1-2x-x^2)\,y'' + 2(1+x)\,y' - 2y = 0$ given that $y_1 = x+1$ is one solution.

$$y'' + \frac{2(1+x)}{1-2x-x^2}\,y' - \frac{2}{1-2x-x^2}\,y = 0 \qquad (2\text{pts})$$

$$y_2(x) = y_1(x) \int \frac{e^{\int -P(x)\,dx}}{(y_1(x))^2}\,dx \qquad (2\text{pts})$$

$$= (x+1) \int \frac{e^{\int \frac{-2-2x}{1-2x-x^2}\,dx}}{x^2+2x+1}\,dx \qquad (1\text{pts})$$

$$= (x+1) \int \frac{1-2x-x^2}{x^2+2x+1}\,dx \qquad (2\text{pts})$$

$$= -(x+1) \int \frac{x^2+2x-1}{x^2+2x+1}\,dx$$

$$= -(x+1) \int 1 - \frac{2}{(x+1)^2}\,dx \qquad (2\text{pts})$$

$$= -(x+1)\left(x + \frac{2}{x+1}\right) \qquad (2\text{pts})$$

$$= -x^2 - x - 2$$

$\therefore$ The general soln. is

$$y = C_1(x+1) + C_2(x^2+x+2) \qquad (1\text{ pt})$$

9. **(14 points)** Given the differential equation $xdx + (x^2y + 4y)\,dy = 0$.

a) Show that the differential equation is not exact.

$M(x,y) = x \Rightarrow M_y = 0$ (1pt)

$N(x,y) = x^2y + 4y \Rightarrow N_x = 2xy$ (1pt)

Since $M_y \neq N_x$ (2pts)

$\Rightarrow$ the D.E is Not exact.

b) Find an integrating factor that makes the differential equation exact.

$$\frac{M_y - N_x}{N} = \frac{-2xy}{y(x^2+4)} = \frac{-2x}{x^2+4} \quad (2\text{ pts})$$

an integrating factor is

$$u(x) = e^{\int \frac{-2x}{x^2+4}dx} = \frac{1}{x^2+4} \quad (2\text{pts})$$

c) Solve the new Equation.

multiply the D.E by $u(x)$, to get

$$\frac{x}{x^2+4}dx + y\,dy = 0 \quad (1\text{ pt})$$

$M_y = N_x = 0$ (Exact)

So, there is a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = \frac{x}{x^2+4}, \quad \frac{\partial f}{\partial y} = y \quad (2\text{pts})$$

$$f(x,y) = \int y\,dy = \frac{y^2}{2} + g(x) \quad (1\text{ pt})$$

$$\frac{\partial f}{\partial x} = g'(x) = \frac{x}{x^2+4} \Rightarrow g(x) = \frac{1}{2}\ln(x^2+4) \quad (1\text{pt})$$

The Solution is $y^2 + \ln(x^2+4) = C$ (1pt)

10. **(10 points)** Determine the form of a particular solution of $y'' + 4y = \cos^2 x$ by using the undetermined coefficients method.
(Do not evaluate the coefficients).

$y'' + 4y = 0$

$m^2 + 4 = 0 \Rightarrow m = \pm 2i$ (1 pt)

$\therefore y_c = c_1 \cos 2x + c_2 \sin 2x$ (1 pt)

an annihilator of $\cos^2 x = \dfrac{1+\cos 2x}{2}$ is (1 pt)

$A(D) = D(D^2+4)$ (2 pts)

Apply $A(D)$ to both sides of the D.E,

$D(D^2+4)^2 y = 0$

$m(m^2+4)^2 = 0 \Rightarrow m = 0,\ m = \pm 2i,\ \pm 2i$ (1 pts)

$\therefore y = c_1 + c_2 \cos 2x + c_3 x \cos 2x + c_4 \sin 2x + c_5 x \sin 2x$

(2 pts)

y_c

$\therefore y_p = A + Bx\cos 2x + Cx\sin 2x$ (2 pts)

11. **(10 points)** Without solving the system, show that
$X = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$ is the general solution of the homogeneous system $X' = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} X$.

$$X_1 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix} \Rightarrow X_1' = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}$$

$$A X_1 = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix} = X_1' \quad (2\text{pts})$$

$$X_2 = \begin{pmatrix} e^{2t} \\ e^{2t} \\ 0 \end{pmatrix} \Rightarrow X_2' = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \\ 0 \end{pmatrix}$$

$$A X_2 = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} \\ e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \\ 0 \end{pmatrix} = X_2' \quad (2\text{pts})$$

$$X_3 = \begin{pmatrix} e^{2t} \\ 0 \\ e^{2t} \end{pmatrix} \Rightarrow X_3' = \begin{pmatrix} 2e^{2t} \\ 0 \\ 2e^{2t} \end{pmatrix}$$

$$A X_3 = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} \\ 0 \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 0 \\ 2e^{2t} \end{pmatrix} = X_3' \quad (2\text{pts})$$

So, X_1, X_2 and X_3 are solution of the system.

Now, we need to check they are Linearly Independent

$$(2\text{pts}) \quad W(X_1, X_2, X_3) = \begin{vmatrix} e^t & e^{2t} & e^{2t} \\ e^t & e^{2t} & 0 \\ e^t & 0 & e^{2t} \end{vmatrix} = e^t \begin{vmatrix} e^{2t} & e^{2t} \\ e^{2t} & 0 \end{vmatrix} + e^{2t} \begin{vmatrix} e^t & e^{2t} \\ e^t & e^{2t} \end{vmatrix}$$

$$(2\text{ pts}) \quad = -e^{3t} \neq 0$$

So, the form a fundamental set of solutions of the homogeneous system.

So, $X = c_1 X_1 + c_2 X_2 + c_3 X_3$ is the general soln.

12. **(10 points)** Solve the given differential equation by using an appropriate substitution. $\dfrac{dy}{dx} = \tan^2(x+y)$.

let $u = x+y$ (2 pts)

$$\frac{du}{dx} = 1 + \frac{dy}{dx} \quad (1\text{ pt})$$

$$\frac{du}{dx} - 1 = \tan^2 u \quad (1\text{ pt})$$

$$\Rightarrow \frac{du}{dx} = 1 + \tan^2 u = \sec^2 u$$

$$\Rightarrow \cos^2 u \, du = dx \quad (1\text{ pt})$$

$$\Rightarrow \frac{1+\cos 2u}{2} du = dx \quad (1\text{ pt})$$

$$\Rightarrow \frac{1}{2}u + \frac{1}{4}\sin 2u = x + C_1 \quad (2\text{ pts})$$

$$\Rightarrow 2u + \sin 2u = 4x + C$$

$$\Longrightarrow 2x + 2y + \sin 2(x+y) = 4x + C$$

$$\Longrightarrow 2y - 2x + \sin 2(x+y) = C \quad (2\text{ pts})$$