King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 202 - Exam II - Term 143

Duration: 120 minutes

Name: Key ID Number: ____________

Section Number: ____________ Serial Number: ____________

Class Time: ____________ Instructor's Name: ____________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 9 pages of problems (Total of 9 Problems)

Question Number	Points	Maximum Points
1		12
2		10
3		8
4		12
5		10
6		10
7		14
8		10
9		14
Total		100

1. **(12 points)** Solve the IVP:

$$y^{(3)} + y'' + 9y' + 9y = 0$$

subject to $y(0) = 1$, $y'(0) = 7/2$, $y''(0) = 5$.

The auxiliary Equation is $m^3 + m^2 + 9m + 9 = 0$ (1 pt)

$\Rightarrow m^2(m+1) + 9(m+1) = 0$

$\Rightarrow (m+1)(m^2+9) = 0$

$\Rightarrow m = -1, \pm 3i$ (1 pt)

$\therefore$ The general solution is given by

$y = C_1 e^{-x} + C_2 \cos(3x) + C_3 \sin(3x)$ (1 pt)

$y' = -C_1 e^{-x} - 3C_2 \sin(3x) + 3C_3 \cos(3x)$ (1 pt)

$y'' = C_1 e^{-x} - 9C_2 \cos(3x) - 9C_3 \sin(3x)$ (1 pt)

$y(0) = 1 \Rightarrow C_1 + C_2 = 1$ ... (1) (1 pt)

$y'(0) = \frac{7}{2} \Rightarrow -C_1 + 3C_3 = \frac{7}{2}$ ... (2) (1 pt)

$y''(0) = 5 \Rightarrow C_1 - 9C_2 = 5$ ... (3) (1 pt)

Solving for C_1, C_2 and C_3 we get

$C_1 = \frac{7}{5}$ (1 pt)

$C_2 = -\frac{2}{5}$ (1 pt)

$C_3 = \frac{49}{30}$ (1 pt)

So, $y = \frac{7}{5} e^{-x} - \frac{2}{5} \cos(3x) + \frac{49}{30} \sin(3x)$ (1 pt)

2. **(10 points)** Determine the form of a particular solution for

$$y'' + y' + y = x\,\sin x$$

by using the method of undetermined coefficients
(Do not evaluate the constants).

The auxiliary equation for the associated homogeneous equation is $m^2+m+1=0 \Rightarrow m = \frac{-1\pm\sqrt{3}i}{2}$. (2 pts)

$\therefore\ y_c = c_1 e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right)$ (2 pts)

Apply the differential operator $(D^2+1)^2$ to both sides of the given D.E, we get (2 pts)

$(D^2+1)^2(D^2+D+1)\,y = 0 \quad \cdots (1)$

The auxiliary equation of (1) is

$(m^2+1)^2(m^2+m+1)=0$

$\Rightarrow\ m = \pm i \text{ (order 2)},\ \frac{-1\pm\sqrt{3}i}{2}$.

Thus (2 pts)

$$y = \underbrace{c_1 e^{-\frac{1}{2}x}\cos\left(\tfrac{\sqrt{3}}{2}x\right) + c_2 e^{-\frac{1}{2}x}\sin\left(\tfrac{\sqrt{3}}{2}x\right)}_{y_c} + \underbrace{(c_3+c_4x)\cos x + (c_5+c_6x)\sin x}_{y_p}$$

$\therefore\ y_p$ has the form

$y_p = (A+Bx)\cos x + (C+Dx)\sin x$ (2 pts)

3. (**~~12~~ 8 points**) Find the annihilator with lowest degree of

$$5e^{-3x} + 3x^4 + 6x^3e^{3x} + \sqrt{3}x - 9 + xe^{3x}\cos 5x.$$

$$D^5\left(3x^4 + \sqrt{3}x - 9\right) = 0$$

$$(D+3)\left(5e^{-3x}\right) = 0$$

$$(D-3)^4\left(6x^3e^{3x}\right) = 0$$

$$\left((D-3)^2+25\right)^2\left(x e^{3x}\cos 5x\right) = 0$$

So, the annihilator with lowest degree of $5e^{-3x} + 3x^4 + 6x^3e^{3x} + \sqrt{3}x - 9 + xe^{3x}\cos(5x)$ is

$$\underbrace{D^5}_{(2pts)}\ \underbrace{(D+3)}_{(2pts)}\ \underbrace{(D-3)^4}_{(2pts)}\ \underbrace{\left(D^2-6D+34\right)^2}_{(2pts)}$$

4. **(12 points)** Use variation of parameters to solve the DE

$$x^2y'' - 6xy' + 12y = \frac{6x^5}{x^2+1}$$

given that $y_1 = x^3$ and $y_2 = x^4$ are solutions of the corresponding homogeneous DE on $(0, \infty)$.

Assume $y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$

$$W = \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix} = 4x^6 - 3x^6 = x^6 \quad (2\text{ pts})$$

$$W_1 = \begin{vmatrix} 0 & x^4 \\ \frac{6x^3}{x^2+1} & 4x^3 \end{vmatrix} = \frac{-6x^7}{x^2+1} \quad (2\text{ pts})$$

$$W_2 = \begin{vmatrix} x^3 & 0 \\ 3x^2 & \frac{6x^3}{x^2+1} \end{vmatrix} = \frac{6x^6}{x^2+1} \quad (2\text{ pts})$$

$$u_1(x) = \int \frac{W_1}{W}\, dx = \int \frac{-6x}{x^2+1}\, dx = -3\ln(x^2+1) \quad (2\text{pts})$$

$$u_2(x) = \int \frac{W_2}{W}\, dx = \int \frac{6}{x^2+1}\, dx = 6\tan^{-1}(x) \quad (2\text{ pts})$$

$$\therefore y_p = -3x^3\ln(x^2+1) + 6x^4\tan^{-1}(x)$$

and the general solution is

$$y = c_1x^3 + c_2x^4 - 3x^3\ln(x^2+1) + 6x^4\tan^{-1}(x) \quad (2\text{pts})$$

5. **(10 points)** If $y_1 = \sin(x^2)$ is a solution of $xy'' - y' + 4x^3y = 0$, for $x > 0$. Find the general solution of the differential equation.

We use the Reduction of order formula

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx \quad (2 \text{ pts})$$

To get $P(x)$, we write the equation in standard form

$$y'' - \frac{1}{x}y' + 4x^2y = 0 \quad (1 \text{ pt})$$

$$\therefore P(x) = -\frac{1}{x} \quad (1 \text{ pt})$$

$$\Rightarrow y_2(x) = \sin(x^2) \int \frac{e^{\int \frac{1}{x}dx}}{\sin^2(x^2)} dx \quad (1 \text{ pt})$$

$$= \sin(x^2) \int x \csc^2(x^2)\, dx \quad (1 \text{ pt})$$

$$= \sin(x^2) \cdot \left(\frac{-1}{2} \cot(x^2)\right) \quad (2 \text{ pts})$$

$$= -\frac{1}{2} \cos(x^2) \cdot (1 \text{ pt})$$

So, the general solution is

$$y = C_1 \sin(x^2) + C_2 \cos(x^2) \quad (2 \text{ pts})$$

6. **(10 points)** Solve the differential equation

$$x^3y^{(3)} - 6x^2y'' + 19xy' - 27y = 0.$$

let $y = x^m$

$y' = m\,x^{m-1}$

$y'' = m(m-1)\,x^{m-2}$ (2pts)

$y''' = m(m-1)(m-2)\,x^{m-3}$

Substituting in the D.E, we get

$m(m-1)(m-2) - 6m(m-1) + 19m - 27 = 0$

$\Rightarrow m^3 - 3m^2 + 2m - 6m^2 + 6m + 19m - 27 = 0$

$\Rightarrow m^3 - 9m^2 + 27m - 27 = 0$ (2pts)

$\Rightarrow (m-3)^3 = 0 \Rightarrow m = 3$ (order 3) (2pts)

So, the general solution is

$$y = \left[\underbrace{c_1}_{(1pt)} + \underbrace{c_2(\ln x)}_{(1pt)} + \underbrace{c_3(\ln x)^2}_{(1pt)}\right] \underbrace{x^3}_{(1pt)}, \quad x > 0.$$

7. **(14 points)** Find two power series solutions of the equation

$$y'' + x^2 y = 0$$

about the ordinary point $x = 0$.

$X=0$ is an ordinary point, the D.E has No singular point So the series solutions converge for all X.

let $y = \sum_{n=0}^{\infty} a_n X^n \Rightarrow y' = \sum_{n=1}^{\infty} n a_n X^{n-1} \Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n X^{n-2}$

(2 pts)

Substituting in the D.E, to get

$$\sum_{n=2}^{\infty} n(n-1) a_n X^{n-2} + \sum_{n=0}^{\infty} a_n X^{n+2} = 0 \quad (1\,pt)$$

$$\Rightarrow 2a_2 + 6a_3 X + \sum_{n=4}^{\infty} n(n-1) a_n X^{n-2} + \sum_{n=0}^{\infty} a_n X^{n+2} = 0$$

$$\Rightarrow 2a_2 + 6a_3 X + \sum_{n=0}^{\infty} (n+3)(n+4) a_{n+4} X^{n+2} + \sum_{n=0}^{\infty} a_n X^{n+2} = 0$$

$$\Rightarrow 2a_2 + 6a_3 X + \sum_{n=0}^{\infty} \left[(n+3)(n+4) a_{n+4} + a_n \right] X^{n+2} = 0 \quad (2\,pts)$$

(1 pt)

$\Rightarrow a_0, a_1$ are arbitrary constants, $a_2 = 0$, $a_3 = 0$ and

$$a_{n+4} = \frac{-1}{(n+3)(n+4)} a_n, \quad n = 0, 1, 2, 3, \ldots \quad (2\,pts)$$

$$a_4 = \frac{-1}{12} a_0, \quad a_6 = 0, \quad a_8 = \frac{1}{(7)(8)(12)} a_0, \ldots \quad (2\,pts)$$

$$a_5 = -\frac{1}{20} a_1, \quad a_7 = 0, \quad a_9 = \frac{1}{(8)(9)(20)} a_1, \ldots \quad (2\,pts)$$

$$y = a_0 \underbrace{\left[1 - \frac{1}{12} X^4 + \frac{1}{(7)(8)(12)} X^8 + \ldots \right]}_{y_1(x)} + a_1 \underbrace{\left[X - \frac{1}{20} X^5 + \frac{1}{(8)(9)(20)} X^9 + \ldots \right]}_{y_2(x)}$$

(2 pts)

8. **(10 points)** Determine the singular points of the differential equation

$$x^3(x^2-36)(x-3)^2y'' + 3x(x-3)y' + (x+6)y = 0$$

and classify them as regular or irregular.

Singular points:

$x^3(x^2-36)(x-3)^2 = 0 \Rightarrow x = 0, 6, -6$ and 3. (3pts)

We write the D.E in standard form

$$y'' + \underbrace{\frac{3}{x^2(x-6)(x+6)(x-3)}}_{P(x)} y' + \underbrace{\frac{1}{x^3(x-6)(x+6)(x-3)^2}}_{Q(x)} y = 0$$

(2pts)

$x=0$ is irregular singular point since (2pts)

$x\,P(x)$ is not analytic at $x=0$.

$x = 6, -6$ and 3 are regular singular point

Since

$$\left.\begin{matrix}(x-6)^2\\(x+6)^2\\(x-3)^2\end{matrix}\right\} Q(x) \text{ and } \left.\begin{matrix}(x-6)\\(x+6)\\(x-3)\end{matrix}\right\} P(x) \text{ are}$$ (3pts)

analytic at $x = 6, -6$ and 3 resp.

9. **(14 points)** Given a Cauchy-Euler equation

$$x^2y'' - 7xy' = \ln x - 7y, \quad x > 0. \ (1)$$

(a) Use a suitable substitution to transfer equation (1) into an equation with constant coefficients.

(b) Use the new equation in part (a) to find the general solution of equation (1).

a) let $x = e^t$ or $t = \ln x$

$x\frac{dy}{dx} = \frac{dy}{dt}$, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$ (2pts)

Substituting in the D.E,

$y''(t) - y'(t) - 7y'(t) = t - 7y(t)$

$\Rightarrow y'' - 8y' + 7y = t \quad \cdots (1)$ (2pts)

The auxiliary Eqn. of $y'' - 8y' + 7y = 0$ is

$m^2 - 8m + 7 = 0 \Rightarrow (m-1)(m-7) = 0$ (1pt)

$\Rightarrow m = 1, 7$

$y_c = c_1 e^t + c_2 e^{7t}$ (2pts)

b) We apply D^2 to both sides of (1), to get

$D^2(D^2 - 8D + 7)y = 0$ (1pt)

$\therefore y_p = A + Bt$, $y_p' = B$, $y_p'' = 0$ (2pts)

$\Rightarrow -8B + 7A + 7Bt = t$

$\Rightarrow B = \frac{1}{7}$, $A = \frac{8}{49}$ (2pts)

(2pts)

$\therefore y = c_1 e^t + c_2 e^{7t} + \frac{1}{7}t + \frac{8}{49} \Rightarrow y(x) = c_1 x + c_2 x^7 + \frac{1}{7}\ln x + \frac{8}{49}$