King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 202 - Exam I - Term 143

Duration: 120 minutes

Name: Key

ID Number: ____________________

Section Number: ____________________

Serial Number: ____________________

Class Time: ____________________

Instructor's Name: ____________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 10 pages of problems (Total of 10 Problems)

Question Number	Points	Maximum Points
1		8
2		8
3		10
4		10
5		8
6		12
7		12
8		12
9		8
10		12
Total		100

1. **(8 points)** Verify that $x(t) = c_1 \cos t + c_2 \sin t + \frac{1}{2} t \sin t$ is a solution of the differential equation $\frac{d^2x}{dt^2} + x = \cos t$.

$X(t) = C_1 \cos t + C_2 \sin t + \frac{1}{2} t \sin t$

$\Rightarrow X'(t) = -C_1 \sin t + C_2 \cos t + \frac{1}{2} \sin t + \frac{1}{2} t \cos t$ (3pts)

$\Rightarrow X''(t) = -C_1 \cos t - C_2 \sin t + \frac{1}{2} \cos t + \frac{1}{2} \cos t - \frac{1}{2} t \sin t$

$= -C_1 \cos t - C_2 \sin t + \cos t - \frac{1}{2} t \sin t$ (3pts)

Substituting in the D.E, we get

$L.H.S = X'' + X$

$= -\cancel{C_1 \cos t} - \cancel{C_2 \sin t} + \cos t - \cancel{\frac{1}{2} t \sin t} + \cancel{C_1 \cos t} + \cancel{C_2 \sin t}$
$+ \cancel{\frac{1}{2} t \sin t}$ (2 pts)

$= \cos t$

$= R.H.S$

Hence, $X(t) = C_1 \cos t + C_2 \sin t + \frac{1}{2} t \sin t$

is a solution of the D.E.

2. **(8 points)** Show that the Initial Value Problem

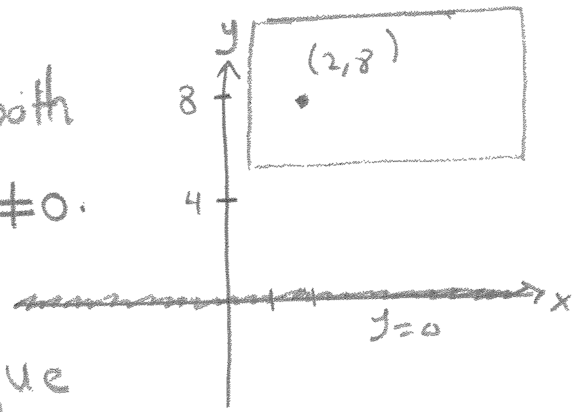
$$y' = \sqrt{x}\, y^{2/3},\ y(2) = 8,$$

has a unique solution.

$f(x,y) = \sqrt{x}\ y^{\frac{2}{3}}$ which is Continous for all (x, y) such that $x \geq 0$. (2 pts)

$\frac{\partial f}{\partial y} = \frac{2}{3}\sqrt{x}\ y^{-\frac{1}{3}}$ which is Continuous for all (x, y) such that $x \geq 0$ and $y \neq 0$. (2 pts)

(1 pt) So, $f(x,y)$ and $\frac{\partial f}{\partial y}$ are both Continous if $x \geq 0$ and $y \neq 0$.



So, the I.V.P has a unique Solution in any rectangle Containing $(2,8)$ in its interior with $x \geq 0$ and $y \neq 0$. (3 pts)

Say, $R = \{(x,y) \mid x \geq 0,\ y > 0\}$.

3. **(10 points)** Solve the seperable differential equation $xy^3\,dx + e^{-x}\,(y+1)\,dy = 0$.

$$xy^3\,dx + e^{-x}(y+1)\,dy = 0$$

$$\Rightarrow\ xe^{x}\,dx + \frac{y+1}{y^3}\,dy = 0 \quad , \ y \neq 0 \,.\ (3\text{ pts})$$

$$\Rightarrow\ xe^{x}\,dx + \left(y^{-2} + y^{-3}\right)dy = 0$$

$$\Rightarrow\ \int xe^{x}\,dx + \int\left(y^{-2} + y^{-3}\right)dy = C$$

$$\Rightarrow\ xe^{x} - e^{x} - \frac{1}{y} - \frac{1}{2y^2} = C \quad 1\text{ pt}$$

(3 pts) (3 pts)

4. **(10 points)** Solve the homogeneous differential equation

$$(y + \sqrt{xy})\, dx - x dy = 0, \quad x > 0.$$

Let $y = ux$ (1 pt)

$\Rightarrow dy = u\,dx + x\,du$ (2 pts)

substituting in the D.E, gives

$$\left(ux + \sqrt{ux^2}\right) dx - x\,(u\,dx + x\,du) = 0$$

$$\Rightarrow \left(ux + x\sqrt{u}\right) dx - x\,(u\,dx + x\,du) = 0$$

$$\Rightarrow \left(u + \sqrt{u}\right) dx - u\,dx - x\,du = 0$$

$$\Rightarrow \sqrt{u}\, dx - x\, du = 0 \quad (3\text{pts})$$

$$\Rightarrow \frac{dx}{x} - \frac{du}{\sqrt{u}} = 0 \quad (1\text{ pt})$$

$$\Rightarrow \ln x - 2\sqrt{u} = c \quad (x>0) \quad (2\text{pts})$$

$$\Rightarrow \ln x - 2\sqrt{\frac{y}{x}} = C \quad (1\text{ pt})$$

5. **(8 points)** Find a function $K(x)$ that makes

$$[3xy + xy\,e^x]\,dx + \left[K(x) + \frac{3}{2}x^2\right]\,dy = 0$$

an exact differential equation.

$M(x,y) = 3xy + xy\,e^x$ (1 pt)

$N(x,y) = K(x) + \frac{3}{2}x^2$ (1 pt)

the D.E is Exact $\iff M_y = N_x$ (1 pt)

$M_y = 3x + x\,e^x$ (1 pt)

$N_x = K'(x) + 3x$ (1 pt)

$M_y = N_x \Rightarrow K'(x) = x\,e^x$ (1 pt)

$\Rightarrow K(x) = x\,e^x - e^x + C$ (2 pts)

6. **(12 points)** Solve the initial value problem

$$(2x + y^2 e^{xy})\, dx + e^{xy}(1 + xy)\, dy = 0, \; y(0) = 4.$$

$$\left.\begin{aligned} M(x,y) &= 2x + y^2 e^{xy} \\ \Rightarrow M_y &= 2y e^{xy} + x y^2 e^{xy} \end{aligned}\right\} \text{ 2 pts}$$

$$\text{2pts} \left\{\begin{aligned} N(x,y) &= e^{xy} + xy e^{xy} \\ \Rightarrow N_x &= y e^{xy} + y e^{xy} + x y^2 e^{xy} = 2y e^{xy} + x y^2 e^{xy} \end{aligned}\right.$$

Since $M_y = N_x$, so ~~it~~ it is an Exact D.E.

$\therefore$ there is a function $f(x,y)$ such that

$$\left.\frac{\partial f}{\partial x} = 2x + y^2 e^{xy} \quad \text{and} \quad \frac{\partial f}{\partial y} = e^{xy} + xy e^{xy}\right\} \text{ (1 pt)}$$

$$f(x,y) = \int \frac{\partial f}{\partial x}\, dx = \int (2x + y^2 e^{xy})\, dx$$

$$= x^2 + y e^{xy} + g(y) \quad \text{(2 pts)}$$

$$\frac{\partial f}{\partial y} = \cancel{e^{xy}} + \cancel{xy e^{xy}} + g'(y) = \cancel{e^{xy}} + \cancel{xy e^{xy}}$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = c_1 \quad \text{(2 pts)}$$

$$\therefore f(x,y) = x^2 + y e^{xy} + c_1$$

So, the solution of the D.E is $f(x,y) = c_2$

or

$$x^2 + y e^{xy} = c \quad \text{(1 pt)}$$

$y(0) = 4 \Rightarrow c = 4$. Hence, the solution is $x^2 + y e^{xy} = 4$

(2 pts)

7. **(12 points)** Reduce the given differential equation to a linear differential equation, then solve the new equation:

$$xy' - 2y = 12x^3 y^{1/2}.$$

$$y' - \frac{2}{x} y = 12 x^2 y^{\frac{1}{2}} \quad (1\text{ pt})$$

So, it is of Bernoulli's type with $n = \frac{1}{2}$

let $u = y^{\frac{1}{2}}$ (2 pts)

$$\Rightarrow \frac{du}{dx} = \frac{1}{2} y^{-\frac{1}{2}} \frac{dy}{dx}$$

$$y^{-\frac{1}{2}} \frac{dy}{dx} - \frac{2}{x} y^{\frac{1}{2}} = 12 x^2$$

$$\Rightarrow 2\frac{du}{dx} - \frac{2}{x} u = 12 x^2 \quad (2\text{ pts})$$

$$\Rightarrow \frac{du}{dx} - \frac{1}{x} u = 6x^2 \quad \text{---} \ (*) \quad (1\text{ pt})$$

(2 pts) an integrating factor is $e^{\int -\frac{1}{x} dx} = \frac{1}{x}$, $x>0$

multiplying the equation $(*)$ by $\frac{1}{x}$, we get

(2 pts) $\frac{d}{dx}\left(\frac{u}{x}\right) = 6x$

$$\Rightarrow \frac{u}{x} = 3x^2 + C$$

(1 pt) $\Rightarrow u = 3x^3 + Cx$ or

(1 pt) $\Rightarrow y^{\frac{1}{2}} = 3x^3 + Cx$ or $y = (Cx + 3x^3)^2$

8. **(12 points)** Radium decomposes at a rate proportional to the amount present at a time t. If 40% of the original amount p_0 disappears in 1400 years, find the amount present after 2800 years.

Let the amount present at a time t be $p(t)$.

then,

$$\frac{dP}{dt} = K\,p(t) \quad \text{with} \quad P(0) = P_0 \text{ and } P(1400) = 0.6\,P_0$$ (3pts)

Solving the D.E, gives

$$P(t) = c\,e^{Kt}$$ (3pts)

$$P(0) = P_0 \Rightarrow C = P_0 \quad \leftarrow (1\,pt)$$

$$\therefore P(t) = P_0\,e^{Kt}$$

$$P(1400) = 0.6\,P_0 \Rightarrow \cancel{P_0}\,e^{1400K} = 0.6\,\cancel{P_0}$$

$$\Rightarrow 1400K = \ln(0.6)$$

$$\Rightarrow K = \frac{1}{1400}\ln(0.6) \quad (2\,pt)$$

$$\therefore P(t) = P_0\,e^{\frac{t}{1400}\ln(0.6)} \quad (1\,pt)$$

The amount present after 2800 Years

is
$$P(2800) = P_0\,e^{2\ln(0.6)}$$
$$= P_0\,e^{\ln(0.6)^2} = 0.36\,P_0 \quad (2pts)$$

9. **(8 points)** Let $y = c_1 e^x \cos x + c_2 e^x \sin x$, be a two-parameter family of solutions of the differential equation $y'' - 2y' + 2y = 0$. Determine whether a member of the family can be found that satisfies the boundary conditions $y(0) = 1$ and $y'(\pi) = 0$.

$y(0) = 1 \Rightarrow c_1 = 1$ (2pts)

$\Rightarrow y = e^x \cos x + c_2 e^x \sin x$

(2pts) $\Rightarrow y' = e^x \cos x - e^x \sin x + c_2 e^x \sin x + c_2 e^x \cos x$

$y'(\pi) = 0 \Rightarrow$

$-e^\pi - c_2 e^\pi = 0$

$\Rightarrow c_2 e^\pi = -e^\pi$

$\Rightarrow c_2 = -1$ (2pts)

$\therefore y = e^x \cos x - e^x \sin x$ (2pts)

10. **(12 points)** Show that $y_1(x) = e^{-3x}$, $y_2(x) = e^{4x}$ form a fundamental set of solutions of the differential equation $y'' - y' - 12y = 0$ on $(-\infty, \infty)$. Form the general solution.

$y'' - y' - 12y = 0$ is 2nd order D.E

We need to check that

(1pt) 1) $y_1(x) = e^{-3x}$ and $y_2(x) = e^{4x}$ are solutions of the D.E

(2pts) $y_1(x) = e^{-3x} \Rightarrow y_1'(x) = -3e^{-3x} \Rightarrow y_1'' = 9e^{-3x}$

substituting in the D.E, we get

$L.H.S = y'' - y' - 12y$

(2pts) $= 9e^{-3x} + 3e^{-3x} - 12e^{-3x} = 0 = R.H.S$

$\therefore$ $y_1(x)$ is a solution.

$y_2(x) = e^{4x} \Rightarrow y_2'(x) = 4e^{4x} \Rightarrow y_2''(x) = 16e^{4x}$

(2pts) $L.H.S = y'' - y' - 12y = 16e^{4x} - 4e^{4x} - 12e^{4x}$

$= 0 = R.H.S.$

$\therefore$ $y_2(x)$ is a Solution.

(1pt) 2) They are Linearly Independent

(3 Pts) $W(e^{-3x}, e^{4x}) = \begin{vmatrix} e^{-3x} & e^{4x} \\ -3e^{-3x} & 4e^{4x} \end{vmatrix} = 4e^{x} + 3e^{x}$

$= 7e^{x} \neq 0$ for all $x \in (-\infty, \infty)$.

$\therefore$ They are L.I.

From 1) and 2) They form a fundamental set of Solutions

(1pt) $\rightarrow$ and the general Solution is $y = c_1 e^{-3x} + c_2 e^{4x}$.