King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 201 Major Exam I
The Third Semester of 2014-2015 (143)
Time Allowed: 120 Minutes

Name: ______________________ ID#: ______________

Section/Instructor: ______________ Serial #: ______________

- Mobiles and calculators are not allowed in this exam.
- Provide all necessary steps required in the solution.

Question #	Marks	Maximum Marks
1		10
2		14
3		12
4		14
5		14
6		12
7		14
8		10
Total		100

Q:1 Consider the parametric equations $x = 2 + \sin t,\ y = \cos t + 1$.

(a) (5 points) Eliminate the parameter to find a cartesian equation.

(b) (5 points) Sketch the curve for $0 \le t \le \pi$ and mark the direction in which the curve is traced as t increases.

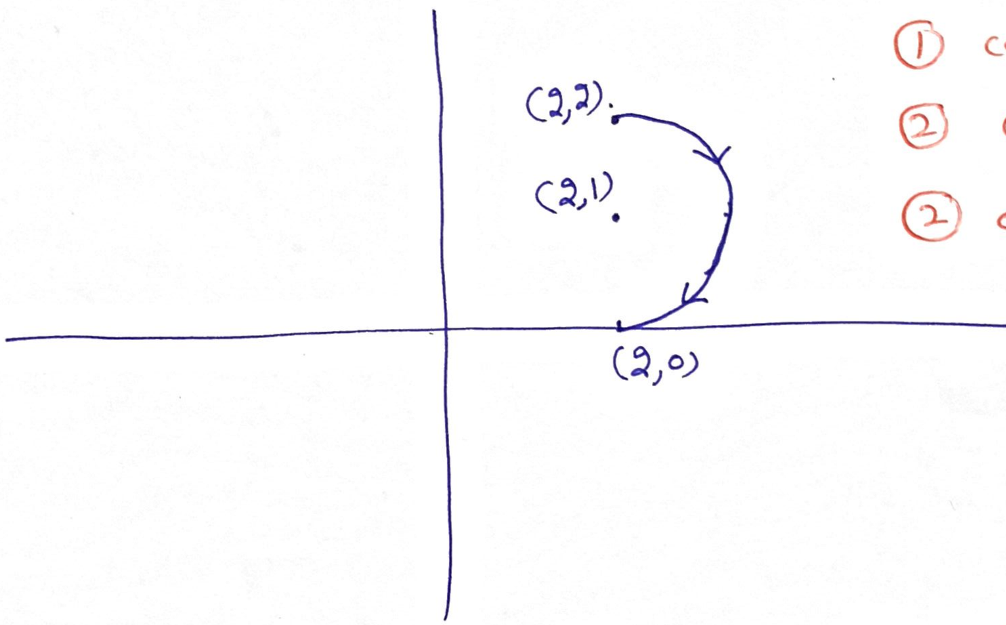
Eliminating t, we obtain

$$(x-2)^2 + (y-1)^2 = \sin^2 t + \cos^2 t = 1$$ 5

1 centre

2 curve

2 direction



The curve of the circle centered at point (2,1) with radious 1 from the point (2,2) to the point (2,0) clockwise direction.

Q:2 (a)(7 points) Find the equation of tangent line to the curve $t = ln\ (x - t)$, $y = t\ e^t$ at $t = 0$.

$t = \ln(x - t)$

$e^t = x - t \Rightarrow x = e^t + t$

$\frac{dx}{dt} = e^t + 1$ ①

$y = t e^t$

$\frac{dy}{dt} = t e^t + e^t$ ①

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t (t+1)}{e^t + 1}$ ①

$\left.\frac{dy}{dx}\right|_{t=0} = \frac{1}{2}$ ①

For $t = 0$, $x = 1$, $y = 0$. ①

Equation of tangent line

$y - 0 = \frac{1}{2}(x - 1)$

or $y = \frac{1}{2}x - \frac{1}{2}$ 02

(b) (7 points) Find the length of the curve
$x = 2t - 2\sin t,\quad y = 2 - 2\cos t,\quad 0 \le t \le 2\pi.$

$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ ①

$= \int_0^{2\pi} \sqrt{(2 - 2\cos t)^2 + (2\sin t)^2}\, dt$ ①

$= \int_0^{2\pi} \sqrt{4 + 4\cos^2 t - 8\cos t + 4\sin^2 t}\, dt$

$= \int_0^{2\pi} \sqrt{8 - 8\cos t}\, dt = \int_0^{2\pi} 2\sqrt{2 - 2\cos t}\, dt$ ①

$= 2\int_0^{2\pi} \sqrt{4\sin^2 \frac{t}{2}}\, dt$ ②

$= 4\int_0^{2\pi} \left|\sin \frac{t}{2}\right| dt = -4 \cdot 2 \left[\cos \frac{t}{2}\right]_0^{2\pi}$ ①

$= -8\left[\cos \pi - \cos 0\right]$

$= (-8)(-2) = 16$ ①

Q:3 (a)(6 points) Write the polar equation $r = (\ln r - \ln\cos\theta)\csc\theta$ in cartesian coordinates.

$$r = (\ln r - \ln\cos\theta)\csc\theta$$

$$= \left(\ln \frac{r}{\cos\theta}\right)\frac{1}{\sin\theta} \qquad (1)$$

$$r\sin\theta = \ln\frac{r}{\cos\theta} \qquad (1)$$

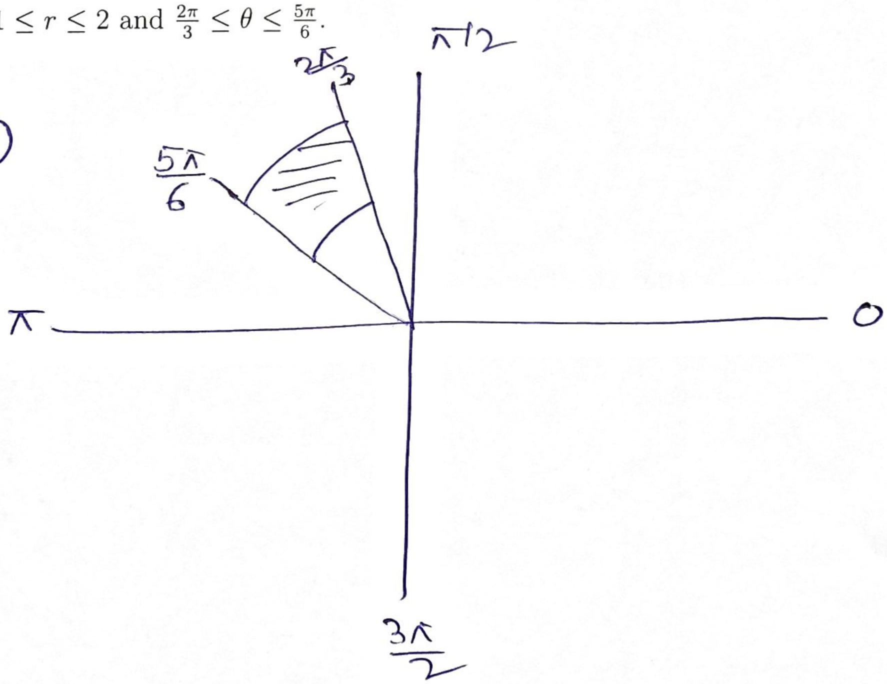
$$y = \ln\frac{r^2}{r\cos\theta} \qquad (1)$$

$$y = \ln\frac{x^2+y^2}{x}, \; x > 0 \qquad (2) + (1)$$

(b) (6 points) Graph the sets of points whose polar coordinates satisfy the following conditions

$$1 \le r \le 2 \text{ and } \frac{2\pi}{3} \le \theta \le \frac{5\pi}{6}.$$

(2) + (2) + (2)



Q:4 (a)(6 points) Identify the symmetries of the curve $r^2 = 4\cos\theta$.

(i) The curve is symmetrical about x-axis because $(r, -\theta)$ on the graph. $r^2 = 4\cos(-\theta)$

$r^2 = 4\cos\theta$ 2

(ii) The curve is symmetrical about origin because, (r, θ) on the graph $\Rightarrow r^2 = 4\cos\theta$ 2

$\Rightarrow (-r)^2 = 4\cos\theta$

$\Rightarrow r^2 = 4\cos\theta \Rightarrow (-r, \theta)$ on the graph.

2 Together, these two symmetries imply symmetry about y-axis.

(b) (8 points) Find the slope of the curve $r = 1 + \sin\theta$ at $\theta = \frac{\pi}{3}$.

$x = r\cos\theta, \quad y = r\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$ 2

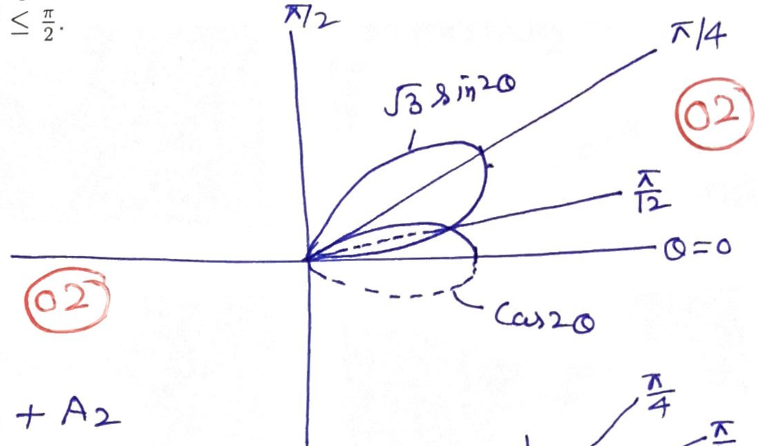
$$= \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$ 2

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\frac{1}{2}\frac{\sqrt{3}}{2} + \left(1+\frac{\sqrt{3}}{2}\right)\frac{1}{2}}{\frac{1}{4} - \left(1+\frac{\sqrt{3}}{2}\right)\frac{\sqrt{3}}{2}}$$ 2

$$= \frac{\frac{\sqrt{3}}{4} + \frac{1}{2} + \frac{\sqrt{3}}{4}}{\frac{1}{4} - \frac{\sqrt{3}}{2} - \frac{3}{4}} = \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}}{-\frac{1}{2} - \frac{\sqrt{3}}{2}}$$ 1

$= -1.$ 1

Q:5 (14 points) Find the area of the region that lies inside both curves $r = \cos 2\theta$ and $r = \sqrt{3}\ \sin 2\theta$ for $0 \le \theta \le \frac{\pi}{2}$.



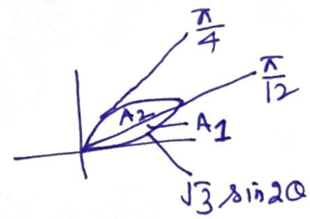
The curve intersect at

$\sqrt{3}\sin 2\theta = \cos 2\theta$

$\tan 2\theta = \frac{1}{\sqrt{3}}$

$2\theta = \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{12}$ 02

Area of the region $= A_1 + A_2$



$A_1 = \frac{1}{2}\int_0^{\pi/12} r^2\, d\theta = \frac{3}{2}\int_0^{\pi/12} \sin^2 2\theta\, d\theta$ 1 + $\frac{1}{2}$

$= \frac{3}{4}\int_0^{\pi/12} (1-\cos 4\theta)\, d\theta$ 1

$= \frac{3}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/12} = \frac{3}{4}\left[\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right]$ 2

$A_2 = \int_{\pi/12}^{\pi/4} \frac{1}{2}(\cos 2\theta)^2\, d\theta$ 1 + $\frac{1}{2}$

$= \int_{\pi/12}^{\pi/4} \left(\frac{1+\cos 4\theta}{4}\right) d\theta$ 1

$= \frac{1}{4}\left[\theta + \frac{\sin 4\theta}{4}\right]_{\pi/12}^{\pi/4}$

$= \frac{1}{4}\left[\frac{\pi}{4} - \frac{\pi}{12} - \frac{\sqrt{3}}{8}\right] = \frac{\pi}{24} - \frac{\sqrt{3}}{32}$ 2

Area $= \frac{3\pi}{48} + \frac{\pi}{24} - \frac{4\sqrt{3}}{32}$

$= \frac{5\pi}{48} - \frac{\sqrt{3}}{8}$ 1

Q:6 (a) (6 points) Find an equation of the sphere that passes through the point $(2, -4, 3)$ and has center $(1, 2, 5)$. Describe the intersection of this sphere with the xz-plane.

Radious $= \sqrt{(1-2)^2 + (2+4)^2 + (5-3)^2}$

$= \sqrt{1+36+4} = \sqrt{41}$ 1

Equation of sphere is

$$(x-1)^2 + (y-2)^2 + (z-5)^2 = 41$$ 2

The intersection of this sphere with xz-plane is the set of all points on the sphere where y-coordinate is zero.

Putting $y=0$ in the equation of sphere, we get

$$(x-1)^2 + (z-5)^2 = 37,$$ 1

Which represents a circle in the xz-plane with Centre $(1, 0, 5)$ and radious $\sqrt{37}$.

1 1

(b) (6 points) If the angle between two **unit vectors** $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$, then find the value of $|3\vec{a} - 2\vec{b}|$.

$|3\vec{a} - 2\vec{b}|^2 = (3\vec{a} - 2\vec{b}) \cdot (3\vec{a} - 2\vec{b})$ 2

$= 9\,\vec{a}\cdot\vec{a} - 12\,\vec{a}\cdot\vec{b} + 4\,\vec{b}\cdot\vec{b}$ 1

$= 9|\vec{a}|^2 - 12|\vec{a}||\vec{b}|\cos\frac{\pi}{3} + 4|\vec{b}|^2$ 1

$= 9 + 4 - 12\cdot 1\cdot 1\cdot \frac{1}{2}$ 1

$= 9 + 4 - 6$

$= 7$

$|3\vec{a} - 2\vec{b}| = \sqrt{7}$ 1

Q:7 (6 points) Find the vector projection of $\vec{a} = <1,1,1>$ onto $\vec{b} = <2,3,4>$ and the scalar component of $\vec{a}$ in the direction of $\vec{b}$.

$$\text{Proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\,\vec{b} \quad (1)$$

$$\vec{a}\cdot\vec{b} = 2+3+4 = 9 \quad (1)$$

$$|\vec{b}|^2 = 4+9+16 = 29 \quad (1)$$

$$\text{Proj}_{\vec{b}}\vec{a} = \frac{9}{29}<2,3,4> \quad (1)$$

Scalar component of $\vec{a}$ in the direction of $\vec{b} = |\vec{a}|\cos\theta$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} \quad (1)$$

$$= \frac{9}{\sqrt{29}} \quad (1)$$

(b) (8 points) Find a unit vector perpendicular to the plane of P(1, -1, 0), Q(2, 1, -1), and R(- 1, 1, 2).

$$\vec{PQ} = \langle 2-1, 1+1, -1-0\rangle = \langle 1,2,-1\rangle \quad (1)$$

$$\vec{PR} = \langle -1-1, 1+1, 2-0\rangle = \langle -2,2,2\rangle \quad (1)$$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} \quad (3)$$

$$= \langle 6, 0, 6\rangle$$

$$|\vec{PQ}\times\vec{PR}| = \sqrt{36+36} = 6\sqrt{2} \quad (1)$$

Unit vector perpendicular to the plane $= \dfrac{\vec{PQ}\times\vec{PR}}{|\vec{PQ}\times\vec{PR}|} \quad (1)$

$$= \left\langle \frac{1}{\sqrt{2}}, 0\ \frac{1}{\sqrt{2}} \right\rangle \quad (1)$$

Q:8 (10 points) Find the volume of the parallelepiped determined by the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ where

$$A(1,0,0),\ B(0,2,0),\ C(0,0,3) \text{ and } D(0,1,3).$$

$\overrightarrow{AB} = \langle 0-1, 2-0, 0-0 \rangle = \langle -1, 2, 0 \rangle$ 2

$\overrightarrow{AC} = \langle 0-1, 0-0, 3-0 \rangle = \langle -1, 0, 3 \rangle$ 2

$\overrightarrow{AD} = \langle 0-1, 1-0, 3-0 \rangle = \langle -1, 1, 3 \rangle$ 2

$$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{vmatrix} -1 & 2 & 0 \\ -1 & 0 & 3 \\ -1 & 1 & 3 \end{vmatrix}$$ 2

$$= -1\begin{vmatrix} 0 & 3 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} -1 & 3 \\ -1 & 3 \end{vmatrix} + 0\begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix}$$

$$= +3 - 2(-3+3)$$

$$= +3$$ 2

Volume $= |\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD})| = |+3| = 3$ units cubed