King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 102
Exam I
Term 143
Wednesday 24/06/2015
Net Time Allowed: 120 minutes

MASTER VERSION

1. If $f(x) = \begin{cases} -2x & x < 0 \\ \sqrt{x} & x \geq 0 \end{cases}$, then $\int_{-1}^{4} f(x)\,dx =$

(a) $\frac{19}{3}$

(b) $\frac{17}{4}$

(c) 2

(d) $-\frac{11}{6}$

(e) $\frac{4}{3}$

$$= \int_{-1}^{0} f(x)\,dx + \int_{0}^{4} f(x)\,dx$$

$$= \int_{-1}^{0} -2x\,dx + \int_{0}^{4} \sqrt{x}\,dx$$

$$= -x^2\Big]_{-1}^{0} + \frac{2}{3}x^{3/2}\Big]_{0}^{4}$$

$$= -(0-1) + \frac{2}{3}(8-0)$$

$$= 1 + \frac{16}{3}$$

$$= \frac{19}{3}$$

2. If f' is continuous on $[1,4]$, $f(1) = 12$, and $\int_{1}^{4} f'(x)dx = 17$, then $f(4) =$

(a) 29

(b) 5

(c) -5

(d) 10

(e) 3

$$\int_{1}^{4} f'(x)\,dx = 17$$

$$\Longrightarrow f(x)\Big]_{1}^{4} = 17$$

$$\Longrightarrow f(4) - f(1) = 17$$

$$\Longrightarrow f(4) - 12 = 17$$

$$\Longrightarrow f(4) = 17 + 12 = 29$$

3. If P is a partition of $[0, \pi]$, then

$$\lim_{\|p\| \to 0} \sum_{k=1}^{n} (c_k + \cos c_k) \triangle x_k = \int_0^{\pi} (x + \cos x)\, dx$$

$$= \frac{1}{2} x^2 \Big]_0^{\pi} + \sin x \Big]_0^{\pi}$$

$$= \frac{1}{2}(\pi^2 - 0) + (\sin \pi - \sin 0)$$

$$= \frac{1}{2}\pi^2 + (0 - 0)$$

$$= \frac{\pi^2}{2}$$

(a) $\dfrac{\pi^2}{2}$

(b) π^2

(c) 2π

(d) $\dfrac{\pi^2}{4}$

(e) 3π

4. If $\int_0^{-3} g(t)\, dt = \sqrt{2}$, then $\int_{-3}^{0} \left(1 + \dfrac{3g(x)}{\sqrt{2}}\right) dx =$

$$= \int_{-3}^{0} 1\, dx + \frac{3}{\sqrt{2}} \int_{-3}^{0} g(x)\, dx$$

$$= x \Big]_{-3}^{0} + \frac{3}{\sqrt{2}}(-\sqrt{2})$$

$$= [0 - (-3)] - 3$$

$$= 3 - 3$$

$$= 0$$

(a) 0

(b) $3 - \sqrt{2}$

(c) -6

(d) $-3 + 2\sqrt{2}$

(e) $1 + \sqrt{2}$

5. $\displaystyle\int \frac{1}{x(\ln x)^3}\,dx =$

(a) $-\dfrac{1}{2(\ln x)^2} + C$

(b) $-\dfrac{1}{3\ln x} + C$

(c) $\dfrac{2}{(\ln x)^2} + C$

(d) $\dfrac{x}{\ln x} + C$

(e) $2\ln(\ln x) + C$

$u = \ln x \implies du = \frac{1}{x}\,dx$

$$\int \frac{1}{x(\ln x)^3}\,dx = \int \frac{1}{u^3}\,du = \int u^{-3}\,du = \frac{u^{-2}}{-2} + C = -\frac{1}{2u^2} + C = -\frac{1}{2(\ln x)^2} + C$$

6. $\displaystyle\int_{-5}^{0} 8\sqrt{25 - x^2}\,dx =$

(a) $50\,\pi$

(b) $\dfrac{25\pi}{4}$

(c) $10\,\pi$

(d) $\dfrac{5\pi}{2}$

(e) $2\,\pi$

$8\displaystyle\int_{-5}^{0} \sqrt{25 - x^2}\,dx$

$= 8 \cdot$ Area of a quarter of a circle of radius 5

$= 8 \cdot \frac{1}{4}\,\pi\, 5^2$

$= 2\,\pi\,(25)$

$= 50\,\pi$

7. Which one of the following statements is **TRUE**? f and g are integrable functions on $(-\infty, \infty)$.

✓ (a) If $f(x) \leq g(x)$ for all x, then $\int_{-1}^{-2} f(x)\,dx \geq \int_{-1}^{-2} g(x)\,dx$

(b) If $\int_1^3 f(x)dx = 3$ and $\int_2^3 f(x)\,dx = 5$, then $\int_1^2 f(x)\,dx = 8$

(c) If $\int_1^3 f(x)\,dx = 3$ then $\int_1^3 (f(x))^2\,dx = 9$

(d) If $f(x) \leq 2$ on $(-\infty, \infty)$, then $\int_0^7 f(x)dx \leq 2$

(e) $-\int_1^3 f(x)\,dx + \int_3^4 f(x)\,dx = \int_1^4 f(x)\,dx$

$f(x) \leq g(x)$ for all x $\Rightarrow \int_{-2}^{-1} f(x)dx \leq \int_{-2}^{-1} g(x)\,dx$

$\Rightarrow -\int_{-1}^{-2} f(x)\,dx \leq -\int_{-1}^{-2} g(x)\,dx$

$\Rightarrow \int_{-1}^{-2} f(x)\,dx \geq \int_{-1}^{-2} g(x)\,dx$

8. The **slope** of the tangent line to the curve

$$y = \int_{2^x}^{1} \sqrt{t}\,dt = -\int_1^{2^x} \sqrt{t}\,dt$$

at $x = 2$ is

$\frac{dy}{dx} = -\sqrt{2^x} \cdot 2^x \ln 2$

slope $= \left.\frac{dy}{dx}\right|_{x=2} = -\sqrt{2^2} \cdot 2^2 \ln 2$

$= -2 \cdot 4 \ln 2$

$= -8 \ln 2$

(a) $-8 \ln 2$

(b) 2

(c) $\sqrt{2}$

(d) $-\frac{14}{3}$

(e) $4 \ln 2$

$y =$

9. If $\int_{\sqrt{x}}^{x^3} \sqrt{t}\,\sin t\,dt$, then $\dfrac{dy}{dx} =$

$\sqrt{x^3}\,\sin(x^3)\cdot 3x^2 - \sqrt{\sqrt{x}}\,\sin(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}$

$= 3\,x^{3/2+2}\,\sin(x^3) - x^{1/4}\,\sin\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$

$= 3\,x^{7/2}\,\sin(x^3) - \frac{\sin\sqrt{x}}{2\,x^{1/4}}$

(a) $3x^{7/2}\sin(x^3) - \dfrac{\sin\sqrt{x}}{2x^{1/4}}$

(b) $\dfrac{\sin\sqrt{x}}{2x^{1/4}} - 3x^{7/2}\sin(x^3)$

(c) $x^{3/2}\sin(x^3) - x^{1/4}\sin\sqrt{x}$

(d) $3x^{3/2}\sin(x^3) + \dfrac{\sin\sqrt{x}}{2x^{1/4}}$

(e) $x\sqrt{x}\sin(x^3) - \sqrt{x}\sin\sqrt{x}$

10. $\int(\cos x - \sin x)^2\,dx =$ $\int \cos^2 x - 2\sin x\cos x + \sin^2 x \quad dx$

$= \int 1 - \sin(2x) \quad dx$

$= x + \frac{1}{2}\cos(2x) + C$

(a) $x + \dfrac{1}{2}\cos(2x) + c$

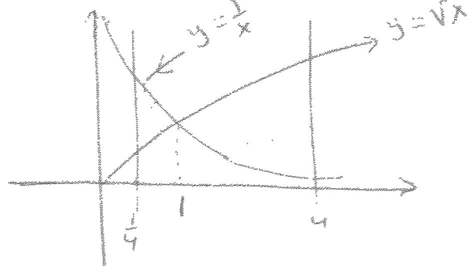
(b) $x + \sin(2x) + c$

(c) $\dfrac{1}{2}x - \cos x + c$

(d) $x + \cos(2x) + c$

(e) $x - \dfrac{1}{2}\sin(2x) + c$

11. The **area** of the region between the curves $y = \sqrt{x}$ and $y = \dfrac{1}{x}$ from $x = \dfrac{1}{4}$ to $x = 4$ is equal to



(a) $\dfrac{49}{12}$

(b) $\dfrac{3}{2} + 2\ln 4$

(c) $\dfrac{11}{2}$

(d) $\dfrac{5}{2} - \ln 4$

(e) $\dfrac{35}{6}$

$$A = \int_{1/4}^{1} \frac{1}{x} - \sqrt{x}\,dx + \int_{1}^{4} \sqrt{x} - \frac{1}{x}\,dx$$

$$= \ln|x| - \frac{2}{3}x^{3/2}\Big]_{1/4}^{1} + \frac{2}{3}x^{3/2} - \ln|x|\Big]_{1}^{4}$$

$$= \left(0 - \frac{2}{3}\right) - \left(-\ln 4 - \frac{2}{3}\cdot\frac{1}{8}\right) + \left(\frac{2}{3}\cdot 8 - \ln 4\right) - \left(\frac{2}{3} - 0\right)$$

$$= -\frac{2}{3} + \ln 4 + \frac{1}{12} + \frac{16}{3} - \ln 4 - \frac{2}{3}$$

$$= \frac{12}{3} + \frac{1}{12}$$

$$= \frac{49}{12}$$

12. The **area of the surface** generated by rotating the curve $y = \dfrac{x^3}{9}, 0 \le x \le 2$, about the x-axis is equal to

$$S = \int_0^2 2\pi \cdot \frac{x^3}{9}\sqrt{1 + \left(\frac{x^2}{3}\right)^2}\,dx$$

$$= \frac{2\pi}{9}\int_0^2 x^3\sqrt{1 + \frac{x^4}{9}}\,dx$$

$$u = 1 + \frac{x^4}{9} \Longrightarrow du = \frac{4}{9}x^3\,dx$$

$$x = 0 \Longrightarrow u = 1$$
$$x = 2 \Longrightarrow u = 1 + \frac{16}{9} = \frac{25}{9}$$

(a) $\dfrac{98}{81}\pi$

(b) $\dfrac{49}{81}\pi$

(c) $\dfrac{25}{3}\pi$

(d) $\dfrac{\pi}{27}$

(e) $\dfrac{5\pi}{9}$

$$= \frac{2\pi}{9}\cdot\frac{9}{4}\int_1^{25/9}\sqrt{u}\,du$$

$$= \frac{\pi}{2}\cdot\frac{2}{3}u^{3/2}\Big]_1^{25/9}$$

$$= \frac{\pi}{3}\left(\left(\frac{5}{3}\right)^3 - 1\right)$$

$$= \frac{\pi}{3}\cdot\frac{125-27}{27} = \frac{\pi}{3}\cdot\frac{98}{27} = \frac{98\pi}{81}$$

13. $\displaystyle\int_{\pi/4}^{\pi/2} \cot\theta \cdot \ln(\sin\theta)\, d\theta =$

$u = \ln(\sin\theta)$

$du = \frac{1}{\sin\theta} \cdot \cos\theta\, d\theta = \cot\theta\, d\theta$

$\theta = \frac{\pi}{4} \Rightarrow u = \ln\left(\frac{1}{\sqrt{2}}\right) = -\ln\sqrt{2} = -\frac{1}{2}\ln 2$

$\theta = \frac{\pi}{2} \Rightarrow u = \ln 1 = 0$

(a) $-\dfrac{(\ln 2)^2}{8}$

(b) $\dfrac{\ln 2}{2}$

(c) $-\dfrac{\ln 2}{2}$

(d) 0

(e) $\dfrac{\pi}{8}$

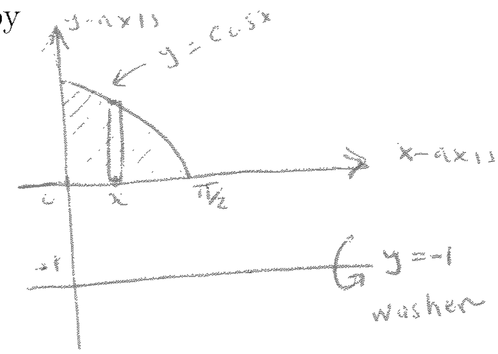
$$\int_{-\frac{1}{2}\ln 2}^{0} u\, du = \frac{1}{2}u^2\Big]_{-\frac{1}{2}\ln 2}^{0}$$

$$= \frac{1}{2}\left(0 - \left(\frac{\ln 2}{2}\right)^2\right)$$

$$= \frac{1}{2} \cdot -\frac{(\ln 2)^2}{4}$$

$$= -\frac{(\ln 2)^2}{8}$$

14. The **volume** of the solid generated by revolving the region bounded by the curves $y = \cos x$ and $y = 0$, for $0 \le x \le \frac{\pi}{2}$, about the line $y = -1$ is given by



(a) $\displaystyle\int_0^{\pi/2} \pi\left[(1+\cos x)^2 - 1\right] dx$

(b) $\displaystyle\int_0^{\pi/2} \pi\left[\cos^2 x - 1\right] dx$

(c) $\displaystyle\int_0^{\pi/2} \pi \cos^2 x\, dx$

(d) $\displaystyle\int_0^{\pi/2} \pi\left[1 - (1-\cos x)^2\right] dx$

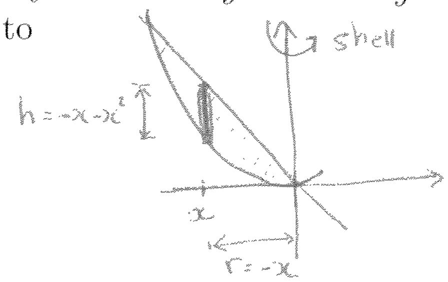
(e) $\displaystyle\int_0^{\pi/2} \pi(1+\cos x)\, dx$

outer radius = $\cos x + 1$

inner radius = 1

$$V = \int_0^{\pi/2} \pi\left[(\cos x + 1)^2 - 1^2\right] dx$$

15. The **volume** of the solid generated by rotating the region enclosed by the curves $y = x^2$ and $y = -x$ about the y-axis is equal to



(a) $\frac{\pi}{6}$

(b) $\frac{\pi}{3}$

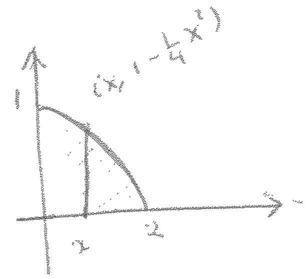
(c) $\frac{\pi}{2}$

(d) π

(e) 2π

$$V = \int_{-1}^{0} 2\pi \cdot -x \cdot (-x - x^2)\, dx$$
$$= 2\pi \int_{-1}^{0} x^2 + x^3 \, dx$$
$$= 2\pi \cdot \left. \frac{1}{3}x^3 + \frac{1}{4}x^4 \right]_{-1}^{0}$$
$$= 2\pi \cdot \left[0 - \left(-\frac{1}{3} + \frac{1}{4}\right)\right]$$
$$= 2\pi \cdot \frac{1}{12}$$
$$= \frac{\pi}{6}$$

16. The base of a solid lies in the <u>**first quadrant**</u> and bounded by the parabola $y = 1 - \frac{1}{4}x^2$, the x- axis, and the y-axis. If the cross sections perpendicular to the x-axis are **squares**, then the volume of the solid is equal to



$$A(x) = \left(1 - \frac{1}{4}x^2\right)^2$$

(a) $\frac{16}{15}$

(b) $\frac{8}{15}$

(c) $\frac{14}{15}$

(d) $\frac{11}{15}$

(e) $\frac{17}{15}$

$$V = \int_{0}^{2} A(x)\, dx$$
$$= \int_{0}^{2} 1 - \frac{1}{2}x^2 + \frac{1}{16}x^4 \, dx$$
$$= \left. x - \frac{x^3}{6} + \frac{x^5}{80} \right]_{0}^{2}$$
$$= 2 - \frac{8}{6} + \frac{32}{80}$$
$$= 2 - \frac{4}{3} + \frac{2}{5} = \frac{30 - 20 + 6}{15} = \frac{16}{15}$$

17. If $\int_5^{15} f(2x)\,dx = 3$, then $\int_2^6 f(5x)\,dx =$

Let $5x = 2u$. Then

$\cdot\ 5\,dx = 2\,du \implies dx = \frac{2}{5}\,du$

$\cdot\ x = \cancel{2} \implies 10 = 2u \implies u = 5$

$\cdot\ x = 6 \implies 30 = 2u \implies u = 15$

$\int_2^6 f(5x)\,dx = \int_5^{15} f(2u)\cdot\frac{2}{5}\,du$

$= \frac{2}{5}\int_5^{15} f(2u)\,du$

$= \frac{2}{5}\cdot 3 = \frac{6}{5}$

(a) $\frac{6}{5}$

(b) -1

(c) 1

(d) 3

(e) $\frac{3}{4}$

18. $\int_0^{\ln\sqrt{3}} \frac{1}{e^x + e^{-x}}\,dx = \int_0^{\ln\sqrt{3}} \frac{e^x}{e^{2x}+1}\,dx$, $u = e^x \implies du = e^x\,dx$

$x = 0 \implies u = 1$

$x = \ln\sqrt{3} \implies u = \sqrt{3}$

$= \int_1^{\sqrt{3}} \frac{1}{u^2+1}\,du$

$= \tan^{-1} u \Big]_1^{\sqrt{3}}$

$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$

$= \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{4\pi - 3\pi}{12} = \frac{\pi}{12}$

(a) $\frac{\pi}{12}$

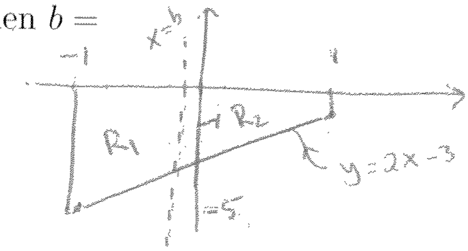
(b) $\frac{1}{2}\pi\ln 3$

(c) $\frac{5\pi}{12}$

(d) $\frac{7\pi}{12}$

(e) $\frac{\pi}{2}$

19. The line $x = b$ divides the region bounded by the curves $y = 2x - 3$, $y = 0$, $x = -1$, $x = 1$ into two regions R_1 and R_2 such that R_1 lies to the left of the line $x = b$. If the area of R_2 is **one third** the area of R_1, then $b =$



(a) $\frac{3-\sqrt{7}}{2}$

(b) $\frac{6-\sqrt{5}}{3}$

(c) $\frac{1}{3}$

(d) $-\frac{1}{3}$

(e) $\frac{6+\sqrt{5}}{3}$

area $(R_2) = \frac{1}{3}$ area (R_1)

$-\int_b^1 (2x-3)\,dx = \frac{1}{3}\cdot -\int_{-1}^{b} (2x-3)\,dx$

$-\left(x^2-3x)\right]_b^1 = \frac{-1}{3}\left(x^2-3x)\right]_{-1}^{b}$

$-\left[-2-(b^2-3b)\right] = -\frac{1}{3}\left(b^2-3b-4\right)$

$-2-b^2+3b = \frac{1}{3}\left(b^2-3b-4\right)$

$-6-3b^2+9b = b^2-3b-4$

$4b^2-12b+2=0$

$2b^2-6b+1=0$

$b = \frac{6\pm\sqrt{36-8}}{4} = \frac{6\pm\sqrt{28}}{4} = \frac{3\pm\sqrt{7}}{2}$

$\Longrightarrow b = \frac{3-\sqrt{7}}{2}$ since $\frac{3+\sqrt{7}}{2} > 1$

20. The **length** of the curve $y = \frac{1}{6}x^3 + \frac{1}{2x}$, $1 \le x \le 2$, is equal to

(a) $\frac{17}{12}$

(b) $\frac{7}{4}$

(c) $\frac{5}{2}$

(d) $\frac{9}{4}$

(e) $\frac{11}{3}$

$y' = \frac{x^2}{2} - \frac{1}{2x^2}$

$(y')^2 = \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}$

$1+(y')^2 = \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$

$\sqrt{1+(y')^2} = \frac{x^2}{2} + \frac{1}{2x^2}$

$L = \int_1^2 \sqrt{1+(y')^2}\,dx = \int_1^2 \frac{1}{2}x^2 + \frac{1}{2x^2}\,dx$

$= \left.\frac{1}{6}x^3 - \frac{1}{2x}\right]_1^2$

$= \left(\frac{8}{6} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{2}\right)$

$= \frac{7}{6} + \frac{1}{4} = \frac{34}{24} = \frac{17}{12}$