King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 102 - Exam II - Term 143

Duration: 120 minutes

Solution KEY

Name:______________________ ID Number:______________________

Section Number:______________________ Serial Number:______________________

Class Time: ______________________ Instructor's Name:______________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have **5 pages** of problems (Total of **9** Problems)

Question Number	Points	Maximum Points
1		8
2		16
3		10
4		8
5		10
6		14
7		14
8		8
9		12
Total		100

1. **(8 points)** Find $\int_1^2 \frac{3^{\log_2 x}}{x}\,dx$.

Let $u = \log_2 x$ ①. Then. $du = \frac{1}{x\ln 2}\,dx$ ①

. $x = 1 \Longrightarrow u = 0$ ①

. $x = 2 \Longrightarrow u = 1$ ①

. $\int_1^2 \frac{3^{\log_2 x}}{x}\,dx = \ln 2 \int_0^1 3^u\,du$ ①

$= \ln 2 \cdot \frac{3^u}{\ln 3}\Big]_0^1$ ②

$= \frac{\ln 2}{\ln 3}(3-1) = \frac{2\ln 2}{\ln 3}$ ①

2. a) **(8 points)** If $x < 0$ and $\cosh^2 x = \frac{36}{25}$, then find $\tanh x$ and $\sinh(2x)$.

• $\cosh^2 x = \frac{36}{25} \Longrightarrow \cosh x = \frac{6}{5}$ ① (as $\cosh x > 0$ for all x) ①

• $\cosh^2 x - \sinh^2 x = 1$ ① $\Longrightarrow \sinh^2 x = \frac{36}{25} - 1 = \frac{11}{25}$ ①

$\Longrightarrow \sinh x = -\frac{\sqrt{11}}{5}$ ① (as $\sinh x < 0$ when $x < 0$)

. $\tanh x = \frac{\sinh x}{\cosh x} = \frac{-\sqrt{11}/5}{6/5} = -\frac{\sqrt{11}}{6}$ ①

. $\sinh(2x) = 2\sinh x \cosh x$ ①

$= 2 \cdot \frac{-\sqrt{11}}{5} \cdot \frac{6}{5} = -\frac{12\sqrt{11}}{25}$ ①

b) **(8 points)** Let $f(x) = \frac{1+\sinh x}{1-\sinh x}$. Find $f'(\ln 2)$. Write your answer in the form $\frac{p}{q}$, where p and q are nonzero integers.

. $f'(x) = \frac{(1-\sinh x)\cosh x - (1+\sinh x)(-\cosh x)}{(1-\sinh x)^2} = \frac{2\cosh x}{(1-\sinh x)^2}$ ②

. $f'(\ln 2) = \frac{2\cosh(\ln 2)}{(1-\sinh(\ln 2))^2}$

$= \frac{2\cdot\frac{5}{4}}{\left(1-\frac{3}{4}\right)^2}$

$= 40$ ②

. $\cosh(\ln 2) = \frac{e^{\ln 2} + e^{-\ln 2}}{2}$ ①

$= \frac{1}{2}\left(2 + \frac{1}{2}\right) = \frac{5}{4}$ ①

. $\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2}$ ①

$= \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{3}{4}$ ①

3. **(10 points)** Evaluate $\int \cos(2\ln x)\,dx$, $= I$

Use integration by parts:

$u = \cos(2\ln x) \qquad dv = dx$

$du = \frac{-2\sin(2\ln x)}{x}\,dx\,, \quad v = x$

$I = x\cos(2\ln x) + 2\int \sin(2\ln x)\,dx$ (4)

$\downarrow$ by Parts again

$u = \sin(2\ln x) \qquad dv = dx$

$du = \frac{2\cos(2\ln x)}{x}\,dx\,, \qquad v = x$

$I = x\cos(2\ln x) + 2\left[x\sin(2\ln x) - 2\int \cos(2\ln x)\,dx\right]$ (3)

$I = x\cos(2\ln x) + 2x\sin(2\ln x) - 4I$ (1)

$5I = x\cos(2\ln x) + 2x\sin(2\ln x)$ (1)

$I = \frac{x}{5}\left[\cos(2\ln x) + 2\sin(2\ln x)\right] + C$ (1)

4. **(8 points)** Evaluate $\int \sec^4(x)\,dx$,

$\int \sec^4 x\,dx = \int \sec^2 x \cdot \sec^2 x\,dx$ (1)

$= \int (1+\tan^2 x)\cdot \sec^2 x\,dx$ (2)

$\downarrow$ Let $u = \tan x$. Then $du = \sec^2 x\,dx$ (1)

$= \int (1+u^2)\,du$ (2)

$= u + \frac{1}{3}u^3 + C$ (1)

$= \tan x + \frac{1}{3}\tan^3 x + C$ (1)

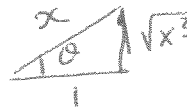
5. **(10 points)** Evaluate $\int \frac{\sqrt{x^2-1}}{x}\,dx,\ x>1.$

Let $x = \sec\theta$ ①, $0 \le \theta < \frac{\pi}{2}$ ① (as $x>1$).

Then $dx = \sec\theta\tan\theta\,d\theta$ ①

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$ since $0 \le \theta < \frac{\pi}{2}$ ①

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\tan\theta}{\sec\theta}\cdot \sec\theta\tan\theta\,d\theta = \int \tan^2\theta\,d\theta \quad ①$$



$$= \int(\sec^2\theta - 1)\,d\theta \quad ②$$

$$= \tan\theta - \theta + C \quad ①$$

$$= \sqrt{x^2-1} - \sec^{-1}(x) + C \quad ②$$

6. **(14 points)** Evaluate $\int \frac{3x^3+4x-2}{x^4+2x^2}\,dx$

④ = 1+1+2

- $\frac{3x^3+4x-2}{x^2(x^2+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2}$

$\Longrightarrow 3x^3+4x-2 = Ax(x^2+2) + B(x^2+2) + (Cx+D)x^2$

- $x=0 \Longrightarrow -2 = 2B \Longrightarrow B=-1$
- Coff of x : $4 = 2A \Longrightarrow A = 2$
- Coff of x^2 : $0 = B + D = -1 + D \Longrightarrow D = 1$
- Coff of x^3 : $3 = A + C = 2 + C \Longrightarrow C = 1$

④. 1 pt for each constant

$$I = \int \frac{2}{x} - \frac{1}{x^2} + \frac{x+1}{x^2+2}\,dx$$

$$= \int \frac{2}{x} - \frac{1}{x^2} + \frac{x}{x^2+2} + \frac{1}{x^2+2}\,dx \quad ①$$

$$= 2\ln|x| + \frac{1}{x} + \frac{1}{2}\ln(x^2+2) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

① ① ① ②

7. Evaluate the improper integral, or show that it diverges.

a) **(6 points)** $\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$

$$\int_2^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-} \int_2^t (3-x)^{-1/2}\,dx \quad ②$$

$$= \lim_{t\to 3^-} -2(3-x)^{1/2}\Big]_2^t \quad ②$$

$$= \lim_{t\to 3^-} -2\sqrt{3-t} + 2 \quad ①$$

$$= -2(0) + 2$$

$$= 2 \quad ①$$

b) **(8 points)** $\int_0^\infty \tanh x\,dx = \lim_{t\to\infty} \int_0^t \tanh x\,dx$ ②

8

$$= \lim_{t\to\infty} \ln|\cosh x|\Big]_0^t \quad ②$$

$$= \lim_{t\to\infty} \ln|\cosh t| - \ln 1$$

$$= \lim_{t\to\infty} \ln|\cosh t| \quad ①$$

$$= \infty \quad ②$$

The integral diverges. ①

8. **(8 points)** Determine whether the sequence $\left\{\frac{(\ln n)^2}{\sqrt{n}}\right\}_{n=1}^{\infty}$ is convergent or divergent. If it is convergent, find its limit.

$$\lim_{n\to\infty} \frac{(\ln n)^2}{\sqrt{n}} = \lim_{n\to\infty} \frac{2\ln n/n}{1/2\sqrt{n}} \quad \text{L'hospital's Rule}$$

②

$$= \lim_{n\to\infty} 4\cdot\frac{\ln n}{\sqrt{n}}$$

$$= \lim_{n\to\infty} 4\cdot\frac{1/n}{1/2\sqrt{n}} \quad \text{, L'hospital's Rule}$$

$$= \lim_{n\to\infty} 8\cdot\frac{1}{\sqrt{n}} \quad ②$$

$$= 8\cdot 0 = 0 \quad ②$$

The seq is convergent & its limit is 0. ②

9. **(12 points)** Evaluate $\int \frac{\cos x}{(2-\cos^2 x)\sin x}\,dx$.

$$I = \int \frac{\cos x}{(2-1+\sin^2 x)\sin x}\,dx$$

$$= \int \frac{\cos x}{(1+\sin^2 x)\sin x}\,dx \quad 3$$

Let $u = \sin x$. Then $du = \cos x\,dx$

$$= \int \frac{1}{u(u^2+1)}\,du \quad 2$$

<u>Decompose</u>: $\frac{1}{u(u^2+1)} = \frac{A}{u} + \frac{Bu+C}{u^2+1}$ $\boxed{1}$

$$\Longrightarrow \quad 1 = A(u^2+1) + u(Bu+C)$$
$$= (A+B)u^2 + Cu + A$$
$$\Longrightarrow \quad A = 1,\ C = 0,\ A+B = 0$$
$$\Longrightarrow \quad A = 1,\ B = -1,\ C = 0$$

$\boxed{1+1+1}$

$$= \int \frac{1}{u} - \frac{u}{u^2+1}\,du \quad 4$$

$$= \ln|u| - \frac{1}{2}\ln(u^2+1) + C \quad 2$$

$$= \ln|\sin x| - \frac{1}{2}\ln(\sin^2 x + 1) + C \quad 1$$