8.2.1. Let X have the pmf $f(x;\theta) = \theta^x(1-\theta)^{1-x}$, $x = 0, 1$, zero elsewhere. We test the simple hypothesis $H_0 : \theta = \frac{1}{4}$ against the alternative composite hypothesis $H_1 : \theta < \frac{1}{4}$ by taking a random sample of size 10 and rejecting $H_0 : \theta = \frac{1}{4}$ if and only if the observed values $x_1, x_2, \ldots, x_{10}$ of the sample observations are such that $\sum_1^{10} x_i \le 1$. Find the power function $\gamma(\theta)$, $0 < \theta \le \frac{1}{4}$, of this test.

8.2.2. Let X have a pdf of the form $f(x;\theta) = 1/\theta$, $0 < x < \theta$, zero elsewhere. Let $Y_1 < Y_2 < Y_3 < Y_4$ denote the order statistics of a random sample of size 4 from this distribution. Let the observed value of Y_4 be y_4. We reject $H_0 : \theta = 1$ and accept $H_1 : \theta \ne 1$ if either $y_4 \le \frac{1}{2}$ or $y_4 > 1$. Find the power function $\gamma(\theta)$, $0 < \theta$, of the test.

8.2.3. Consider a normal distribution of the form $N(\theta, 4)$. The simple hypothesis $H_0 : \theta = 0$ is rejected, and the alternative composite hypothesis $H_1 : \theta > 0$ is accepted if and only if the observed mean $\overline{x}$ of a random sample of size 25 is greater than or equal to $\frac{3}{5}$. Find the power function $\gamma(\theta)$, $0 \le \theta$, of this test.

8.2.4. Consider the distributions $N(\mu_1, 400)$ and $N(\mu_2, 225)$. Let $\theta = \mu_1 - \mu_2$. Let $\overline{x}$ and $\overline{y}$ denote the observed means of two independent random samples, each of size n, from these two distributions. We reject $H_0 : \theta = 0$ and accept $H_1 : \theta > 0$ if and only if $\overline{x} - \overline{y} \ge c$. If $\gamma(\theta)$ is the power function of this test, find n and c so that $\gamma(0) = 0.05$ and $\gamma(10) = 0.90$, approximately.

8.2.5. If in Example 8.2.2 of this section $H_0 : \theta = \theta'$, where θ' is a fixed positive number, and $H_1 : \theta < \theta'$, show that the set $\left\{(x_1, x_2, \ldots, x_n) : \sum_1^n x_i^2 \le c\right\}$ is a uniformly most powerful critical region for testing H_0 against H_1.

8.2.6. If, in Example 8.2.2 of this section, $H_0 : \theta = \theta'$, where θ' is a fixed positive number, and $H_1 : \theta \ne \theta'$, show that there is no uniformly most powerful test for testing H_0 against H_1.

8.2.7. Let $X_1, X_2, \ldots, X_{25}$ denote a random sample of size 25 from a normal distribution $N(\theta, 100)$. Find a uniformly most powerful critical region of size $\alpha = 0.10$ for testing $H_0 : \theta = 75$ against $H_1 : \theta > 75$.

8.2.8. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a normal distribution $N(\theta, 16)$. Find the sample size n and a uniformly most powerful test of $H_0 : \theta = 25$ against $H_1 : \theta < 25$ with power function $\gamma(\theta)$ so that approximately $\gamma(25) = 0.10$ and $\gamma(23) = 0.90$.

8.2.9. Consider a distribution having a pmf of the form $f(x;\theta) = \theta^x(1-\theta)^{1-x}$, $x = 0, 1$, zero elsewhere. Let $H_0 : \theta = \frac{1}{20}$ and $H_1 : \theta > \frac{1}{20}$. Use the central limit theorem to determine the sample size n of a random sample so that a uniformly most powerful test of H_0 against H_1 has a power function $\gamma(\theta)$, with approximately $\gamma(\frac{1}{20}) = 0.05$ and $\gamma(\frac{1}{10}) = 0.90$.

8.2.10. Illustrative Example 8.2.1 of this section dealt with a random sample of size $n = 2$ from a gamma distribution with $\alpha = 1$, $\beta = \theta$. Thus the mgf of the distribution is $(1 - \theta t)^{-1}$, $t < 1/\theta$, $\theta \geq 2$. Let $Z = X_1 + X_2$. Show that Z has a gamma distribution with $\alpha = 2$, $\beta = \theta$. Express the power function $\gamma(\theta)$ of Example 8.2.1 in terms of a single integral. Generalize this for a random sample of size n.

8.2.11. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, zero elsewhere, where $\theta > 0$. Find a sufficient statistic for θ and show that a uniformly most powerful test of $H_0 : \theta = 6$ against $H_1 : \theta < 6$ is based on this statistic.

8.2.12. Let X have the pdf $f(x;\theta) = \theta^x(1-\theta)^{1-x}$, $x = 0, 1$, zero elsewhere. We test $H_0 : \theta = \frac{1}{2}$ against $H_1 : \theta < \frac{1}{2}$ by taking a random sample $X_1, X_2, \ldots, X_5$ of size $n = 5$ and rejecting H_0 if $Y = \sum_{1}^{n} X_i$ is observed to be less than or equal to a constant c.

(a) Show that this is a uniformly most powerful test.

(b) Find the significance level when $c = 1$.

(c) Find the significance level when $c = 0$.

8.2.13. Let $X_1, \ldots, X_n$ denote a random sample from a gamma-type distribution with $\alpha = 2$ and $\beta = \theta$. Let $H_0 : \theta = 1$ and $H_1 : \theta > 1$.

(a) Show that there exists a uniformly most powerful test for H_0 against H_1, determine the statistic Y upon which the test may be based, and indicate the nature of the best critical region.

(b) Find the pdf of the statistic Y in Part (a). If we want a significance level of 0.05, write an equation which can be used to determine the critical region. Let $\gamma(\theta)$, $\theta \geq 1$, be the power function of the test. Express the power function as an integral.

Example 8.2.1. Consider the pdf

$$f(x;\theta) = \begin{cases} \frac{1}{\theta}e^{-x/\theta} & 0 < x < \infty \\ 0 & \text{elsewhere,} \end{cases}$$

of Exercises 8.1.2 and 8.1.3. It is desired to test the simple hypothesis $H_0 : \theta = 2$ against the alternative composite hypothesis $H_1 : \theta > 2$. Thus $\Omega = \{\theta : \theta \geq 2\}$. A random sample, X_1, X_2, of size $n = 2$ will be used, and the critical region is $C = \{(x_1, x_2) : 9.5 \leq x_1 + x_2 < \infty\}$. It was shown in the example cited that the significance level of the test is approximately 0.05 and the power of the test when $\theta = 4$ is approximately 0.31. The power function $\gamma(\theta)$ of the test for all $\theta \geq 2$ will now be obtained. We have

$$\begin{aligned} \gamma(\theta) &= 1 - \int_0^{9.5} \int_0^{9.5 - x_2} \frac{1}{\theta^2} \exp\left(-\frac{x_1 + x_2}{\theta}\right) dx_1 dx_2 \\ &= \left(\frac{\theta + 9.5}{\theta}\right) e^{-9.5/\theta}, \quad 2 \leq \theta. \end{aligned}$$

For example, $\gamma(2) = 0.05$, $\gamma(4) = 0.31$, and $\gamma(9.5) = 2/e \doteq 0.74$. It is shown (Exercise 8.1.3) that the set $C = \{(x_1, x_2) : 9.5 \leq x_1 + x_2 < \infty\}$ is a best critical region of size 0.05 for testing the simple hypothesis $H_0 : \theta = 2$ against each simple hypothesis in the composite hypothesis $H_1 : \theta > 2$. ■