KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DEPARTMENT OF MATHEMATICS & STATISTICS
DHAHRAN, SAUDI ARABIA

STAT 502: Statistical Inference

Semester 142

First Major Exam

Saturday April 04, 2015

6:30 − 9:30 pm

Q.No.1:- (5+5 marks) The distribution of the sample mid-range (R) from a uniform distribution of a random variable X on $\left(\theta - \frac{1}{2}, \theta + \frac{1}{2}\right)$ has the density function:

$$f(r) = n2^{n-1}\left\{\frac{1}{2} - |r-\theta|\right\}^{n-1}; \quad \theta - \frac{1}{2} < r < \theta + \frac{1}{2}$$

(a) Show that the sample mid-range is an unbiased estimator of θ.

(b) Show that the sample mid-range is a consistent estimator of θ.

Q.No.2:- (7 marks) If $\hat{p}_1$ is the most efficient estimator of p and $\hat{p}_2$ is a less efficient estimator with relative efficiency $e = \frac{\mathrm{Var}(\hat{p}_1)}{\mathrm{Var}(\hat{p}_2)}$ and the correlation coefficient between $\hat{p}_1$ and $\hat{p}_2$ is ρ. If we define another estimator as $\hat{p}_3 = \frac{(1-\rho\sqrt{e})\hat{p}_1 + (e-\rho\sqrt{e})\hat{p}_2}{(1+e-2\rho\sqrt{e})}$, then show that $\rho = \sqrt{e}$.

Q.No.3:- (3+3+3 marks) Show that the following densities belong to the exponential family and give the sufficient statistic(s) for the unknown parameter(s).

(a) Negative Binomial distribution

$$f(x;\theta) = \binom{x+r-1}{r-1} p^r (1-p)^x; \quad x = r, r+1, r+2, \ldots \ldots$$

(b) Weibull distribution

$$f(x;\theta) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(x/\lambda\right)^k}; \quad x > 0$$

(c) Beta Distribution

$$f(x;\theta) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}; \quad 0 < x < 1$$

where $\beta(a,b)$ is the beta function defined as $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Q.No.4:- (7 marks) In a sequence of n Bernoulli trials with p as the probability of success, x successes were observed. Show that $\hat{p}\hat{q}^2$ is a biased estimator of pq^2 but the bias $\to 0$ as $n \to \infty$, where $q = 1 - p$, $\hat{p} = \frac{x}{n}$ and $\hat{q} = 1 - \hat{p}$.

Q.No.5:- (6+4 marks)

(a) Suppose $X_1, X_2, \dots \dots, X_n$ form a random sample from the density function $f(x;\theta)$, subject to a number of regularity conditions. Find the Cramer-Rao lower bound for the variance of a biased estimator of $g(\theta)$ where $g(\theta)$ is some function of the unknown parameter θ.

(b) Show that $E\left(\frac{\partial}{\partial\theta}\text{Log}_e L(x;\theta)\right)^2 = -E\left(\frac{\partial^2}{\partial\theta^2}\text{Log}_e L(x;\theta)\right)$

Q.No.6:- (5 marks) If $X_1, X_2, \dots \dots, X_n$ is a random sample from Rayleigh distribution with probability density function $f(x;\theta) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}$; $x > 0$. Find whether there exist any Minimum Variance Bound (MVB) estimator for θ. If yes, find its sampling variance as well.

Q.No.7:- (8+2 marks) Suppose that 2-dimensional vector $(x_1, y_1), (x_2, y_2), \dots \dots, (x_n, y_n)$ form a random sample from a bivariate normal distribution with probability density function:

$$f(x;\theta) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right\}}; \quad \begin{matrix} -\infty < x < \infty \\ -\infty < y < \infty \end{matrix}$$

where μ_1 and μ_2 are the unknown means of X and Y respectively, σ_1^2 and σ_2^2 are the known variances of X and Y respectively and the correlation coefficient (ρ) between X and Y is also known.

(a) Find the Maximum Likelihood Estimators (MLE) of μ_1 and μ_2.

(b) Comment on the 4 basic properties (unbiasedness, consistency, sufficiency, efficiency) of the estimators found in part (a).

Q.No.8:- (7 marks) Suppose that Y follows exponential distribution with probability density function $f(y;\theta) = \theta e^{-\theta y}$; $y > 0$. We observe $y_1 = 3$, $y_2 = 6$, $y_3 = 10$ and the prior of θ is $h(\theta) = \frac{3\theta^2}{19}$; $2 < \theta < 3$. Find the expression for Bayes' estimate of θ.

With the Best Wishes