King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 572, Final Exam
Semester II, 2015 (142)

Name:	
ID :	

Q		Points
1		**50**
2		**50**
3		**50**
4		**50**
5		**80**
Total		**280**

(1) Consider the equation

$$-\Delta u = f,\ \Omega = (0,1)\times(0,1)$$

with

$$u = 0 \quad \text{on} \quad \Gamma_1$$

and the Robin boundary condition

$$\alpha u + \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_2$$

where $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and α is a constant. Derive a variational formulation for this problem and give conditions on α and f (if any are required) that guarantee it has a unique solution in H^1. (Hint: multiply by v and integrate by parts, converting the $\frac{\partial u}{\partial \nu} v$ term to uv using the boundary condition.)

---

(2) Consider the pure advection equation

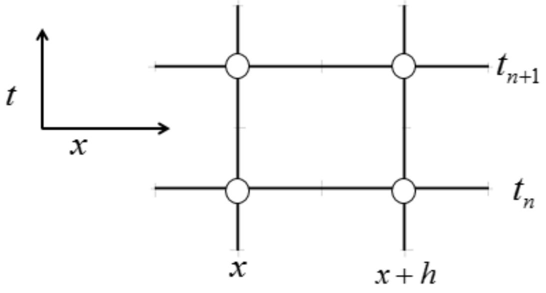
$u_t + cu_x = 0, \ -\infty < x < +\infty, t > 0, c > 0$ (2.1)

subject to

$$u(x,0) = w(x)$$

Discretized using a 4-point stencil

$u_j^n + a_2 u_j^{n+1} + a_3 u_{j+1}^{n+1} + a_4 u_{j+1}^n = 0$ (2.2)



(a) Find the only $o(h^2 + k^2)$ accurate scheme for (1.1) in the form (1.2)

(b) Show that this scheme is unconditionally stable.

---

(3) Consider the equation

$$u_t + cu_x = 0, \; -\infty < x < +\infty, t > 0, c > 0 \quad (2.1)$$

with

$$u(x,0) = w(x)$$

It is easy to show that the solution is given by:

$$u(x,t) = v(x+ct).$$

(a) State the Courant-Friedrichs-Lewy condition (CFL)
(b) Give an example of a scheme which satisfy CFL condition
(c) Give an example of a scheme whch does not satisfy CFL condition

---

(4) Consider the following initial-boundary value problem for *Burger's* equation

Seek $u = u(x,t)$ defined on $[0,1]\times[0,T]$ such that

$$u_t + u\,u_x = \varepsilon u_{xx} \qquad (x,t)\in(0,1)\times(0,T] \text{ where } \varepsilon>0$$
$$u(x,0)=v(x) \qquad x\in(0,1)$$
$$u(0,t)=u(1,t)=0 \qquad t\in(0,T],\; v(0)=v(1)=0$$

Douglas and B.F. Jones proposed the following predictor-corrector method for the solution of nonlinear parabolic equations, which in our case has the form:

Seek $\left\{\hat{U}_j^n\right\}_{j=0:J}^{n=0,N}$, $\left\{U_j^n\right\}_{j=0:J}^{n=0,N}$ such that

Predictor:

$$(1)\begin{cases} \dfrac{2}{k}\left(\hat{U}_j^{n+1}-U_j^n\right)+U_j^n\dfrac{\left(U_{j+1}^n-U_{j-1}^n\right)}{2h}=\dfrac{\varepsilon}{h^2}\left(\hat{U}_{j+1}^{n+1}-2\hat{U}_j^{n+1}+\hat{U}_{j-1}^{n+1}\right), \quad 0\le n\le N-1,\; 1\le j\le J-1 \\ \hat{U}_0^{n+1}=\hat{U}_J^{n+1}=0, \quad 0\le n\le N-1 \end{cases}$$

Corrector:

$$(2)\begin{cases} \dfrac{U_j^{n+1}-U_j^n}{k}+\hat{U}_j^{n+1}\dfrac{1}{4h}\left[\left(U_{j+1}^{n+1}-U_{j-1}^{n+1}\right)+\left(U_{j+1}^n-U_{j-1}^n\right)\right]= \\ \dfrac{\varepsilon}{2h^2}\left[\left(U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1}\right)+\left(U_{j+1}^n-2U_j^n+U_{j-1}^n\right)\right], \quad 0\le n\le N-1,\; 1\le j\le J-1 \\ \hat{U}_0^{n+1}=\hat{U}_J^{n+1}=0, \quad 0\le n\le N-1 \end{cases}$$

With $U_j^0 = v(x_j), \quad 0\le j\le J$

So we predict $\hat{U}_j^{n+}$ by (1) and then we correct U_j^{n+1} by (2). Douglas and Jones showed that for k sufficiently small, $\max\limits_{j,n}\left|U_j^n - u(x_j, nk)\right| = O(h^2+k^2)$.

(a) Verify that for the solution of the above predictor-corrector scheme, one has to solve two linear tridiagonal systems of equations per time step, one for the predictor and one for the corrector. [Hint: write these two system in matrix form then the two coefficient matrices are tridiagonal]

(5) Consider the equation

$$-\nabla \cdot (a\nabla u) + b \cdot \nabla u + u^2 = f, \ \Omega = (0,1)\times(0,1)$$

with

$$u = 0 \quad \text{on } \partial\Omega$$

Where the coefficients $a = a(x,y)$, $b = b(x,y)$ are smooth and such that

$$a(x,y) \geq a_0 > 0, \quad \forall (x,y) \in \Omega$$

(a) Use your knowledge in finite element methods to suggest a weak formulation for this problem.

(b) Suggest a discrete space suitable for this problem [give a reason for selecting this discrete space]

(c) After the discretization, Write the resulting algebraic equations in matrix form (note that the system is nonlinear system of equations)

(d) Suggest a method to solve the nonlinear system in (c).

(e) What properties does the Jacobian matrix in (c) has? Symmetric, positive definite, tridagonal matrix, block-tridaiagonal, diagonally dominant, ……

(f) Suggest a method to solve a linear system with Jacobian matrix in (e) as a coefficient matrix of the system [give a reason for selecting your method]