King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 572, Exam-II
Semester II, 2015 (142)

Name:	
ID :	

Q		Points
1		**40**
2		**40**
3		**40**
4		**40**
Choose one only (5 or 6)		
5		**40**
6		**40**
Total		**200**

(1) Consider the following scheme

$$\frac{U_j^{n+1} - U_j^{n-1}}{2k} = c\,\frac{U_{j+1}^{n} - U_j^{n+1} - U_j^{n-1} + U_{j-1}^{n}}{h^2}$$

which approximate the parabolic equation $u_t = c u_{xx}$

Show that the scheme is of order $O(h^2) + O(k^2) + O\left(\left(\frac{k}{h}\right)^2\right)$

(2) Consider the following scheme

$$(1+\theta)\frac{U_j^{n+1}-U_j^n}{k}-\theta\frac{U_j^n-U_j^{n-1}}{k}=c\frac{U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1}}{h^2}$$

(where $\theta \geq 0$) which approximate the parabolic equation $u_t = cu_{xx}$

Use **Fourier method** to show that the scheme is always stable

(3) Consider the problem

$$-\nabla\cdot(a\nabla u)+cu=f,\ \text{in}\ \Omega$$

$$\tfrac{\partial u}{\partial n}=0\ \text{on}\ \partial\Omega_N,\qquad u=0\ \text{on}\ \partial\Omega_D$$

where $a(x)\geq a_0>0, c(x)\geq c_0>0\quad \partial\Omega=\partial\Omega_D\cup\partial\Omega_N$

Formulate a finite element problem and then prove error estimate.

(4) Consider the following problems

$$\begin{aligned}&(u_t,\varphi)+(u,\varphi)=(f,\varphi)\ \forall\varphi\in H_0^1\\&u(0)=v\end{aligned}\qquad (1)\qquad\qquad \begin{aligned}&(u_{h,t},\chi)+(u_h,\chi)=(f,\chi)\ \forall\chi\in S_h\\&u_h(0)=v_h\end{aligned}\qquad (2)$$

Prove the following theorem

<u>Theorem 10.1</u> Let u_h and u be the solutions of (2) and (1). Then

$$\|u_h(t)-u(t)\|\le\|v_h-v\|+Ch^2\left(\|v\|_2+\int_0^t\|u_t\|_2\,ds\right)\quad\text{for}\quad t\ge 0$$

(5) consider the following " θ -method " $(-1 \le \theta \le 1)$, for the convection diffusion problem $\varepsilon > 0, a(x) > 0$

$$-\varepsilon u_{xx} + a(x)u_x = f(x), \qquad 0 < x < 1, \quad u(0) = u(1) = 0,$$

given by

$$-\varepsilon \frac{U^{n+1} - 2U^n + U^{n-1}}{h^2} + a_n \left[\theta \frac{U^n - U^{n-1}}{h} + (1-\theta) \frac{U^{n+1} - U^n}{h} \right] = f_n$$

Find a condition on $\varepsilon, h, a_n, \theta$ which is necessary and sufficient for the scheme to be of positive scheme. Represent this as a stencil.

(6) Consider the Dirichlet problem (4.9)

$$-\Delta u = f \quad in \quad \Omega = (0,1)\times(0,1), \quad with \quad u = 0 \quad on \quad \partial\Omega$$

with $\qquad f(x,y) = x + y$

Take $h = 1/4,$ and set up the linear system arising from the 5 point approximation [DO NOT SOLVE THE LINEAR SYSTEM]