King Fahd University of Petroleum & Minerals
Department of Math. & Stat.

Math 569 - Final Exam (142)

==================================

Name: ID #

==================================

Problem 1	/10
Problem 2	/10
Problem 3	/5
Problem 4	/10
Problem 5	/10
Total	/45 /37

Problem # 1. (10 marks) In $\mathbb{R}^2$, let

$$u(x,y) = \begin{cases} (x^2+y^2)^{1/4}, & x^2+y^2 \le 1 \\ (x^2+y^2)^{-3/2}, & x^2+y^2 > 1 \end{cases}$$

$a)$ Check that $u \in H^1(\mathbb{R}^2)$.
$b)$ Let $v(x) = u(x,0)$. Check if $v \in H^1(\mathbb{R})$ and what can you conclude?
(Notice that u is continuous in $\mathbb{R}^2$)

Problem # 2. (10 marks) Let Ω be a C^1 bounded domain of $\mathbb{R}^N$. On $H_0^1(\Omega)$,we define the bilinear form

$$a(u,v) = \int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} - c \left(\int_\Omega u \right) \left(\int_\Omega v \right),$$

where $c > 0$ and $a_{ij} \in L^\infty(\Omega)$ satisfying, for some $c_0 > 0$,

$$\sum_{i,j=1}^N a_{ij}(x) \xi_i \xi j \geq c_0 |\xi|^2, \qquad a.e.\ x \in \Omega \text{ and } \forall \xi \in \mathbb{R}^N$$

i) Show that a is bounded and find a condition on c so that a is coercive.
ii) In such a case, show that there exists a unique $u \in H_0^1(\Omega)$ satisfying

$$a(u,v) = \int_\Omega fv, \qquad \forall v \in H_0^1(\Omega),$$

where $f \in L^2(\Omega)$.

Problem # 3. (5 marks) Let Ω be a C^3 bounded domain of $\mathbb{R}^N$. Show that the problem

$$\begin{cases} \Delta^2 u = f & \text{in } L^2(\Omega) \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in H_0^1(\Omega) \cap H^4(\Omega)$.

Hint: You may take $v = \Delta u$.

Problem # 4. (10 marks) Let Ω be a smooth and bounded domain of $\mathbb{R}^N$ and $f \in L^2(\Omega)$.
a. Give a weak formulation, in $H^1(\Omega)$, for the problem

$$\begin{cases} u - \Delta u = f & \text{in } \Omega \\ u + \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega \end{cases}$$

where $\frac{\partial u}{\partial \eta}$ is the outer normal derivative.
b. Discuss the existence and uniqueness of a weak solution.
c. Show if the weak solution $u \in H^2(\Omega)$ then the boundary condition is satisfied "in some sense" .

Problem # 5. (10 marks) Let Ω be a smooth and bounded domain of $\mathbb{R}^2$and $f \in L^2(\Omega)$. Define

$$H_*^1(\Omega) = \{v \in H^1(\Omega) \ / \ \int_\Omega v = 0\}.$$

Suppose that $u \in H_*^1(\Omega)$ satisfies

$$\int_\Omega (\nabla u.\nabla v + uv) = \int_\Omega fv, \ \forall v \in H_*^1(\Omega)$$

a. Show that $u \in C(\Omega)$.
b. If $\int_\Omega f = 0$, use the maximum principle theorem to prove that

$$u(x) \leq \max\{\sup_{\partial\Omega} u, \ \sup_\Omega f \ \}, \ \forall x \in \Omega$$

c. Find the equation satisfied by u.