King Fahd University of Petroleum & Minerals
Department of Math. & Stat.

Math 569 Exam 1 (142) Time: 2:00 H

==================================

Name: ID #

==================================

Problem 1	/10
Problem 2	/3
Problem 3	/3
Problem 4	/11
Problem 5	/8
Total	/35

Problem 1
Let $f \in L^1(\mathbb{R})$ such that

$$\int_{-\infty}^{+\infty} f(t)dt = 0 \text{ and } \int_{0}^{+\infty} f(t)dt \neq 0$$

Define $u_n(x) = nf(nx)$, $x \in I = (-1,1)$.
$i)$ With an appropriate change of variable, show that, for any $\varphi \in C([-1,1])$,

$$\int_{-1}^{1} u_n(x)\varphi(x)dx = \int_{-\infty}^{+\infty} f(t)\left[\varphi(\frac{t}{n}) - \varphi(0)\right]\chi_{[-n,n]}dt + \varphi(0)\int_{-n}^{n} f(t)dt$$

$ii)$ Show that

$$\lim_{n\to+\infty}\int_{-1}^{1} u_n(x)\varphi(x)dx = 0, \ \forall\varphi \in C([-1,1]).$$

$iii)$ Show that (u_n) is bounded in $L^1(I)$.
$iv)$ Show that there is no subsequence (u_{n_k}) which converges weakly to some u in $L^1(I)$.
Hint: You may argue by contradiction to show that $u \equiv 0$ and continue.

Problem 2.

Let $I = (0, +\infty)$ and $g: \mathbb{R} \to \mathbb{R}$ be a C^1 function. If $u \in H_0^1(I)$, show that $v \in H_0^1(I)$, where

$$v(x) = g(u(x)) - g(0).$$

Problem 3

Show that there exist $u_0, u_1 \in L^{p'}((-1,1))$, $1 < p < +\infty$, satisfying

$$\int_{-1}^{1} u_0 v + u_1 v' = v(0) + \int_0^1 v,\ \forall v \in W_0^{1,p}(I).$$

Problem 4.
Let $I = (0,1)$ and define the subspace

$$H_*^1(I) = \{v \in H^1(I) \ : \ \int_I v(x)dx = 0\}$$

$a)$ Show that $H_*^1(I)$ is closed.
$b)$ Show there exists a constant $c_* > 0$, for which

$$||v||_2 \leq c_*||v'||_2, \ \forall v \in H_*^1(I).$$

$c)$ Let $f \in L^2(I)$, show that there exists a unique $u \in H_*^1(I)$ such that

$$\int_0^1 u'v' = \int_0^1 fv, \ \forall v \in H_*^1(I).$$

d) Show that $u \in H^2(I)$ and find the equation satisfied by u.
Hint: Notice that $v = \phi - \int_0^1 \phi \in H_*^1(I)$, $\forall \phi \in H^1(I)$.

Problem 5.
In $I = (0,1)$, given the problem

$$(P) \quad \begin{cases} -u'' + \int_0^1 u(x)dx = f, & \text{in } I \\ u(0) = u(1) = 0 \end{cases},$$

where $f \in L^3(I)$.
a) Show that (P) has a unique weak solution $u \in H_0^1(I)$.
b) Show that $u \in W^{2,3}(I)$