King Fahd University of Petroleum & Minerals
Department of Math. & Stat.

Math 568 - Final Exam (142) Time: 2 hours 30 mns

Thursda, May 21, 2015

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Name: ID #
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Problem 1	/5
Problem 2	/8
Problem 3	/12
Problem 4	/12
Problem 5	/8
Total	/45 /35

Problem # 1. (5 marks) Let $u \in C(\overline{\Omega})$ be the solution of the problem

$$\begin{cases} \Delta u(x,y) = 0, & \text{in } \Omega \\ u(x,y) = y^2, & \text{on } \partial\Omega \end{cases},$$

where $\Omega = \{(x,y) \ / \ x^2 + y^2 < 1\}$

a. Find

$$\max_{\overline{\Omega}} u \text{ and } \min_{\overline{\Omega}} u$$

b. Find $u(0,0)$ and

$$\int\int_{\Omega} u(x,y)dxdy$$

Hint: Do not forget the mean value property

Problem # 2. (8 marks) Given the problem

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = 0, & \text{in } \mathbb{R}^2 \times (0,+\infty) \\ u(x,0) = u_0(x), & \text{in } \Omega \end{cases}$$

where

$$u_0(x) = \begin{cases} x_1^2 + x_2^2, & x_1^2 + x_2^2 \leq 1 \\ 1, & x_1^2 + x_2^2 > 1 \end{cases}$$

Find $u(0,t)$ and $\lim_{t\to+\infty} u(0,t)$

Problem # 3. (12 marks) In a cylindrical body, the temperature distribution is given by

$$(P1)\begin{cases} u_t(r,z,t) - [u_{rr}(r,z,t) + 2\,(u_r(r,z,t)/r) + u_{zz}(r,z,t)] = 0, & \text{in } (0,1)\times(0,\pi)\times(0,+\infty) \\ u(r,0,t) = u(r,\pi,t) = 0, & \text{in } [0,1]\times[0,+\infty) \\ u(1,z,t) = 0, & \text{in } [0,\pi]\times[0,+\infty) \\ u(r,z,0) = 2\,(\sin \pi r \sin 2z)\,/r, & \text{in } (0,1]\times[0,\pi] \end{cases}$$

a. Show that $v = ru$ satisfies

$$(P2)\begin{cases} v_t - (v_{rr} + v_{zz}) = 0, & \text{in } (0,1)\times(0,\pi)\times(0,+\infty) \\ v(r,0,t) = v(r,\pi,t) = 0, & \text{in } [0,1]\times[0,+\infty) \\ v(0,z,t) = v(1,z,t) = 0, & \text{in } [0,\pi]\times[0,+\infty) \\ v(r,z,0) = 2\sin \pi r \sin 2z, & \text{in } (0,1]\times[0,\pi] \end{cases}$$

b. Solve $(P2)$ and then write the solution u of $(P1)$

c. Find the temperature $u(0,z,t)$ at the z-axix.

Problem # 4. (12 marks) Given the initial-boundary value problem

$$(P3)\quad \begin{cases} u_{tt}(x,t) - 4u_{xx}(x,t) = 16x^2 + 4, & \text{in } (0,1)\times(0,+\infty) \\ u_x(0,t) = u(1,t) = 0, & t \geq 0 \\ u(x,0) = -\frac{x^4}{3} - \frac{x2}{2} + \frac{5}{6} + 2\cos(5\pi x/2), & 0 \leq x \leq 1 \\ u_t(x,0) = \pi\cos(\pi x/2), & 0 \leq x \leq 1 \end{cases}$$

a. Let $v = u + f(x)$. Find f so that v satisfies

$$(P4)\quad \begin{cases} v_{tt}(x,t) - 4v_{xx}(x,t) = 0, & \text{in } (0,1)\times(0,+\infty) \\ v_x(0,t) = v(1,t) = 0, & t \geq 0 \\ v(x,0) = v_0(x), & 0 \leq x \leq 1 \\ v_t(x,0) = v_1(x), & 0 \leq x \leq 1 \end{cases}$$

for v_0 and v_1 to be determined.
b. Solve $(P4)$ and write down the solution of $(P3)$

Problem # 5. (8 marks) Solve

$$\begin{cases} u_{tt}(x,t) - 9\Delta u(x,t) = 0, & \mathbb{R}^2 \times (0,+\infty) \\ u(x,0) = 1 + x_1 x_2, \qquad u_t(x,0) = x_2^2, & \text{in } \Omega \end{cases}$$

Use

$$\int_0^1 \frac{r^3 dr}{\sqrt{1-r^2}} = \frac{2}{3} \text{ and } \int_0^1 \frac{r dr}{\sqrt{1-r^2}} = 1$$