King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 513 Final Exam
The Second Semester of 2014-2015 (142)
Time Allowed: 180 Minutes

Name: ______________________ ID#: ______________

Section/Instructor: ______________ Serial #: ______________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

Question #	Marks	Maximum Marks
1		10
2		12
3		10
4		12
5		12
6		12
7		12
Total		80

Q:1 (10 points) Solve by separation of variables

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, \quad t > 0,$$

subject to the boundary conditions

$$u(0,t) = u(L,t) = 0, \; t > 0,$$

and initial conditions for $0 < x < L$,

$$u(x,0) = \begin{cases} x & 0 < x < L/2 \\ L - x & L/2 < x < L, \end{cases}$$

$$u_t(x,0) = 0.$$

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Q:2 (12 points) Solve the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0$$

subject to the following initial and **non-homogeneous** boundary conditions

$$u(x,0) = 0,\ u(0,t) = 2,\ u(1,t) = 3,\ 0 < x < 1, t > 0.$$

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Q:3 (10 points) Use Laplace transform method to solve the wave equation

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \quad t > 0$$

with the boundary and initial conditions

$$u(0,t) = 0, \; u(1,t) = 0, t > 0$$

$$u(x,0) = 0, \; \frac{\partial u}{\partial t}|_{t=0} = sin(\pi x), \; 0 < x < 1.$$

Q:4 (12 points) Solve the Laplace equation by separation of variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < a, 0 < y < b,$$
$$u(x,0) = u(0,y) = 0,$$
$$u(a,y) = 0, u(x,b) = x - a.$$

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Q:5 (12 points) Solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r < 1, 0 < z < 2$$

subject to the boundary conditions

$$u(1, z) = 0, 0 < z < 2$$
$$u(r, 0) = 0, 0 < r < 1$$
$$u(r, 2) = u_0, 0 < r < 1$$

solution $u(r, z)$ is bounded at $r = 0$.

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Q:6 (12 points) Find the steady-state temperature in the sphere of radious C by solving

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0, 0 < r < C, 0 < \theta < \pi$$

$$u(C,\theta) = cos(2\theta), \quad 0 < \theta < \pi.$$

(Hint $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1)$).

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Q:7a (6 points) Find $\mathcal{L}^{-1}\{ln(1+\frac{1}{s^2})\}$.

Q:7b (6 points) Use D'Alembert's formula to solve

$$c^2\,\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = sin\ 3x$$
$$u_t(x,0) = sin\ 2x - sin\ x.$$

Formula Sheet

The Fourier-Bessel series of a function f defined on the interval $(0, L)$ is given by

(i) $$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$

$$c_i = \frac{2}{L^2 J_{n+1}^2(\alpha_i L)} \int_0^L x J_n(\alpha_i x) f(x) dx$$

where the α_i are defined by $J_n(\alpha L) = 0$.

(ii) $$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$

$$c_i = \frac{2\alpha_i^2}{()\alpha_i^2 L^2 - n^2 + h^2) J_n^2(\alpha_i L)} \int_0^L x J_n(\alpha_i x) f(x) dx$$

where the α_i are defined by $hJ_n(\alpha L) + \alpha L J_n'(\alpha L) = 0$.

(iii) $$f(x) = c_1 + \sum_{i=2}^{\infty} c_i J_n(\alpha_i x)$$

$$c_1 = \frac{2}{L^2} \int_0^L x f(x) dx, c_i = \frac{2}{L^2 J_0^2(\alpha_i L)} \int_0^L x J_0(\alpha_i x) f(x) dx$$

where the α_i are defined by $J_0'(\alpha L) = 0$.

(a) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$ and $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

(b) If $\mathcal{L}\{f(t)\} = F(s)$ and $a > 0$, then

(1) $\mathcal{L}\{\frac{1}{\sqrt{\pi t}} e^{-\frac{a^2}{4t}}\} = \frac{e^{-a\sqrt{s}}}{\sqrt{s}}$ (2) $\mathcal{L}\{\frac{a}{2\sqrt{\pi t^3}} e^{-\frac{a^2}{4t}}\} = e^{-a\sqrt{s}}$

(3) $\mathcal{L}\{erfc(\frac{a}{2\sqrt{t}})\} = \frac{e^{-a\sqrt{s}}}{s}$

(4) $\mathcal{L}\{e^{ab} e^{b^2 t} \ \ erfc(b\sqrt{t} + \frac{a}{2\sqrt{t}})\} = \frac{e^{-a\sqrt{s}}}{\sqrt{s}(\sqrt{s}+b)}$

(5)$\mathcal{L}\{\frac{2\sqrt{t}}{\sqrt{\pi}} e^{-\frac{a^2}{4t}} - a \ erfc(\frac{a}{2\sqrt{t}})\} = \frac{e^{-a\sqrt{s}}}{s\sqrt{s}}$

(6) $\mathcal{L}\{-e^{ab+b^2 t} \ \ erfc(b\sqrt{t} + \frac{a}{2\sqrt{t}}) + erfc(\frac{a}{2\sqrt{t}})\} = \frac{be^{-a\sqrt{s}}}{s(\sqrt{s}+b)}$