King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 455, Final Exam, Term 142

Part I (75 points)

1. Find the last three decimal digits of 101^{404}. (12 points)
2. Find the remainder when $50!$ is divided by 53. (12 points)
3. If $x \equiv 1 \; mod \; 5^2$ is a solution of the congruence $x^4 + x - 2 \equiv 0 \; mod \; 5^2$, then find the solutions of the congruence $x^4 + x - 2 \equiv 0 \; mod \; 5^3$ that lie above the solution $x \equiv 1 \; mod \; 5^2$. (12 points)
4. Find all primes p for which the congruence $x^2 \equiv 20 \; mod \; p$ is solvable. (15 points)
5. Find all primitive Pythagorean triangles in which the odd leg equals 15. (12 points)
6. Solve $49x^4 + 4 = y^3$ in integers. (12 points)

Part II (75 points)

7. Prove that $105 | n^{25} - n$ for all integers n. (13 points)
8. Prove that $\sigma(n)$ is odd if and only if n is a square or double a square. (14 points)
9. Let p be an odd prime number and a be an integer such that $ord_p(a) = 3$. Prove that $(2a+1)^2 \equiv -3 \; mod \; p$. (14 points)
10. Show that the number of zeros at the end of $(5n)!$ is n more than the number of zeros at the end of $n!$. (14 points)
11. (8+12 points)
    a. Let p be an odd prime. Prove that if there is an integer n such that $p | n^2 - 2$, then $p \equiv 1 \; or \; 7 \; mod \; 8$.
    b. Use part (a), or otherwise, to prove that there are infinitely many primes of the form $8k + 7$. (**Hint:** Assume there are finitely many primes of the given form; call them $p_1, p_2, \cdots, p_n$. Now consider the number $N = (p_1 p_2 \cdots p_n)^2 - 2$.)

All the best,