King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 455, Exam I, Term 142

Part I (50 points)

1. Show that the square of any integer takes one of the following forms $5l$, $5l + 1, or\ 5l + 4$, where l is some integer.
2. Solve the following system in positive integers: $(5x, 3y) = 6, 3x + 2y = 50$.
3. Find all possible values of $\pi(n + 3) - \pi(n)$, where n is a positive integer.
4. Use Fermat's Factorization Method to find nontrivial factors of 5453.
5. Solve the linear equation $373x - 122y = 3$ in integers.

Part II (50 points)

6. Show that $n^2 | (n + 1)^n - 1$ for any positive integer n.
7. Let p_k be the k^{th} prime. Show that $1 + p_1 p_2 \cdots p_n$ is not a perfect square for any positive integer n.
8. Let $n > 2$ be a positive integer. Prove that if one of the two numbers $2^n - 1, 2^n + 1$ is prime, then the other is composite.
9. Prove that if $(a, b) = 1$, then $(a - b, a^3 + b^3) = 1\ or\ 2$.
10. Let M be a positive common multiple of the nonzero integers a and b. Prove that $M = [a, b]$ if and only if $\left(\frac{M}{a}, \frac{M}{b}\right) = 1$.

All the best,

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Solutions

Question # 1: By the division algorithm, any integer n takes one of the following forms: $5k, 5k+1, 5k+2, 5k+3, 5k+4$. Now

$$n = 5k \Rightarrow n^2 = 25k^2 = 5l, where\ l = 5k^2$$

$$n = 5k+1 \Rightarrow n^2 = 25k^2 + 10k + 1 = 5l + 1, where\ l = 5k^2 + 2k$$

$$n = 5k+2 \Rightarrow n^2 = 25k^2 + 20k + 4 = 5l + 4, where\ l = 5k^2 + 4k$$

$$n = 5k+3 \Rightarrow n^2 = 25k^2 + 30k + 9 = 5l + 4, where\ l = 5k^2 + 6k + 1$$

$$n = 5k+4 \Rightarrow n^2 = 25k^2 + 40k + 16 = 5l + 1, where\ l = 5k^2 + 8k + 3$$

We conclude that the square of any integer takes only one of the following forms $5l,\ 5l+1, or\ 5l+4$, where l is some integer.

Question # 2: As $(5x, 3y) = 6$, then $6|5x\ and\ 6|3y$ and hence $6|x\ and\ 2|y$. This implies that $x = 6k\ and\ y = 2l$ for some positive integers $k\ and\ l$. Substituting in the equation $3x + 2y = 50$ and simplifying, we get the equation $9k + 2l = 25$. As $k\ and\ l$ are positive integers, then $25 = 9k + 2l > 9k$. Thus $k < 25/9$ and hence $k = 1\ or\ 2$. The corresponding values of l are 8 and 7/2 respectively. As $k\ and\ l$ are positive integers, then $(k, l) = (1,8)$ and hence $(x, y) = (6,16)$.

Question # 3: Note that $\pi(n+3) - \pi(n)$ equals the number of primes in the interval $(n, n+3]$, or, equivalently, the number of primes in the set $\{n+1, n+2, n+3\}$. As at least one of these three consecutive positive integers is even (and greater than 2), then the number of primes in the set is at most 2. It can be 2 (for example, $\{2, 3, 4\}$), and it can be 1 (for example, $\{4, 5, 6\}$), and it can be zero (for example, $\{8, 9, 10\}$). We conclude that the possible values of $\pi(n+3) - \pi(n)$ are 0, 1, and 2.

Question # 4: As $\sqrt{n} = \sqrt{5453} \approx 73.84$, then we start by taking $x = 74$. Checking the values $x = 74, 75, 76, \cdots$, we see that $x^2 - n$ is a perfect square when $x = 87$ (at the 14th step): $87^2 - n = 2116 = 46^2$. Thus

$$n = 87^2 - 46^2 = (87 - 46)(87 + 46) = 41 \times 133.$$

Noticing that $133 = 7 \times 19$, we get the full prime factorization

$$n = 5453 = 7 \times 19 \times 41.$$

Question # 5: Apply the Euclidean algorithm to find $(373, 122)$:

$$373 = 3(122) + 7$$

$$122 = 17(7) + 3$$

$$7 = 2(3) + 1$$

$$3 = 3(1).$$

Thus $(373, 122) = 1$. Since $1|3$, the given equation is solvable. Solving backward for the remainders, we get

$$1 = 373(35) - 122(107).$$

Multiplying by 3, we get

$$3 = 373(105) - 122(321).$$

Thus one solution of the given equation is given by $(x_0, y_0) = (105, 321)$ and all solutions are given by

$$x = 105 - 122t, \qquad y = 321 - 373t$$

where t is an arbitrary integer.

Question # 6: We use the binomial theorem:

$$(n+1)^n = \sum_{k=0}^{n} \binom{n}{k} n^k = 1 + n^2 + \sum_{k=2}^{n} \binom{n}{k} n^k.$$

Thus we get

$$(n+1)^n - 1 = n^2 + \sum_{k=2}^{n} \binom{n}{k} n^k.$$

Since the first term is clearly divisible by n^2 and each term in the sum is divisible by n^2 (as $n^2|n^k \; for \; k \geq 2$), then $n^2|(n+1)^n - 1$.

<u>Question # 7:</u> Assume that there is a positive integer n and a positive integer x such that

$$1 + p_1 p_2 \cdots p_n = x^2.$$

Since $p_1 = 2$, then $x > 1$ is odd. We write the last equation in the form

$$p_1 p_2 \cdots p_n = (x-1)(x+1).$$

Since $x > 1$ is odd, then $x - 1$ and $x + 1$ are positive integers divisible by 2 and hence $4|(x-1)(x+1)$. This implies that $4|p_1 p_2 \cdots p_n$ and hence $2|p_2 \cdots p_n$ (as $p_1 = 2$), a contradiction (since $p_1, p_2, \cdots, p_n$ are odd primes). We conclude that $1 + p_1 p_2 \cdots p_n$ is not a perfect square for any positive integer n.

<u>Question # 8:</u> Let $n > 2$ be a positive integer. If $2^n - 1$ is prime, then $n = p$ is an odd prime. This implies that $2^n + 1 = 2^p + 1$ is composite as it is divisible by $2 + 1 = 3$ (and $2^p + 1 > 3$.) Next, if $2^n + 1$ is prime, then $n = 2^k$ for some positive integer $k > 1$ (as $n > 2$.) This implies that $2^n - 1 = 2^{2^k} - 1 = (2^{2^{k-1}} - 1)(2^{2^{k-1}} + 1)$. Since both factors are greater than 1 and less than $2^n - 1$, we conclude that $2^n - 1$ is composite.

<u>Question # 9:</u> Let $(a - b, a^3 + b^3) = g$. Then $g|a - b$ and $g|a^3 + b^3$.

Since $a - b|a^3 - b^3$, then we also have $g|a^3 - b^3$. This implies that

$$g|(a^3 + b^3) + (a^3 - b^3) = 2a^3$$

and

$$g|(a^3 + b^3) - (a^3 - b^3) = 2b^3$$

Hence $g|(2a^3, 2b^3)$. But $(2a^3, 2b^3) = 2(a^3, b^3) = 2 \times 1 = 2$ (as $(a, b) = 1$.) Then $g|2$ and hence $g = 1 \; or \; g = 2$. [**Note:** $g = 2$ if both $a \; and \; b$ are odd and $g = 1$ if $a \; and \; b$ have opposite parity.]

<u>**Question # 10:**</u> Without loss of generality, we may assume that $a \; and \; b$ are positive.

Assume first that $M = [a, b]$. Let $\left(\frac{M}{a}, \frac{M}{b}\right) = g$. Then $g|\frac{M}{a}$ and $g|\frac{M}{b}$ and hence $ag|M$ and $bg|M$. This means that M is a positive common multiple of ag and bg and hence $[ag, bg]|M$. As $[ag, bg] = g[a, b] = gM$, then $gM|M$ implies $g|1$ and hence $g = 1$.

Next assume $\left(\frac{M}{a}, \frac{M}{b}\right) = 1$. Then $\left[\frac{M}{a}, \frac{M}{b}\right] = \frac{M}{a}\frac{M}{b}$. (Here we used the fact that $[x, y](x, y) = xy$ when $x \; and \; y$ are positive integers.) Multiplying by ab, we get $[bM. aM] = M^2$. But $[bM, aM] = M[b, a]$. Then we get $M[b, a] = M^2$ and hence $[b, a] = M$.