King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Instructor: Khaled Furati

MATH 411 - Final - Term 142

Duration: 150 minutes

Student Name:

Q#	**Points**	**Points**
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

1. Define the following.

   (a) $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is twice differentiable at $a \in \Omega$.

   (b) Directional derivative of $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ at $a \in \Omega$.

2. Let $R \subset \mathbb{R}^n$ be a rectangle and $f : R \to \mathbb{R}$ be a bounded function. Define the following:

   (a) The Riemann sum, lower Darboux's sum, and upper Darboux's sum of f corresponding to a partition $\mathcal{P}$ of R.

   (b) f is Riemann integrable over R.

   (c) f is Darboux integrable over R.

   (d) $\int_\Omega f$, where $\Omega \subset R$ is a simple set.

3. State the following:

   (a) The equivalence between the Riemann and Darboux definitions of the integral.

   (b) Lebesgue criterion for integrability.

   (c) Change of variables theorem for multiple integrals.

4. State the Fubini's theorem for elementary regions in $\mathbb{R}^2$. Discuss the relation between the existence of the iterated integrals and the integrability of a function. (Hint. $f(x) = \frac{x^2-y^2}{(x^2+y^2)^2}$.)

5. Let $R \subset \mathbb{R}^n$ be a closed rectangle. Show that if $f : R \to \mathbb{R}$ is continuous on R, then f is integrable on R.

6. Let $f : \Omega \to \mathbb{R}$ be integrable on a bounded simple set $\Omega \subset \mathbb{R}^n$. Show that if $f \equiv 0$ almost everywhere then $\int_\Omega f = 0$.

7. Let f be a real-valued differentiable function on an open convex set Ω in $\mathbb{R}^n$. Show that if $\nabla f(x) = 0$ on Ω then f is constant on Ω.

8. Let $I = [0, 1]$, $J = I \cap Q$, $R = I \times I \times I$, and $\Omega = J \times J \times J$. Discuss the integrability on R of the function

$$f(x) = \begin{cases} 1, & x \in \Omega, \\ 0, & x \in R \setminus \Omega. \end{cases}$$

9. Let Ω be a set in $\mathbb{R}^n$ and $f : \Omega \to (0, \infty)$. Let $\{U_k\}$ and $\{V_k\}$ be exhaustions of Ω. Suppose f is integrable over each U_k and each V_k. Show that

$$\lim_{k\to\infty} \int_{U_k} f = \lim_{k\to\infty} \int_{V_k} f.$$

10. Let $f(x, y)$ be a continuous function on $[a, b] \times [c, \infty)$ and $F(x) = \int_c^\infty f(x, y)dy$.

    (a) Define the uniform convergence of the integral $F(x)$.

    (b) State the Weierstrass M-test for uniform convergence.

    (c) Provide sufficient conditions for $F(x)$ to be continuous on $[a, b]$.

    (d) Discuss the continuity of the the function $F(x) = \int_0^\infty 2xye^{-xy^2} dy$ on $[0, 1]$.