King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

Math 345 – Modern Algebra I (Spring 2015)

Final (Duration = 180 minutes)

Student Name________________________________ **ID:** ___________________

Ex #	Score	Maximum
1		20
2		20
3		20
4		20
5		20
6		20
7		20
8		20
9		20
Total		**180**

Exercise 1 [20 minutes = 20 points]

Let A_8 denote the alternating group of order 8.

(a) Let $\sigma \in A_8$ with $\sigma = \alpha_1\alpha_2 \dots \alpha_k$ where the α_i's are disjoint cycles. Show that $k = 2\ or\ 4$.

(b) What is the maximum order an element of A_8 can have?

Exercise 2 [20 minutes = 20 points]

Let G denote the Klein group $\left(\cong \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}\right)$. Show that $Aut(G) \cong S_3$.

Exercise 3 [20 minutes = 20 points]

Let G be an abelian (multiplicative) group such that its only subgroups are $\{1\}$ and G. Prove that either $G = \{1\}$ or G is cyclic with prime order.

Exercise 4 [20 minutes = 20 points] [`Justify your answers`]

(a) [05 points] Give an example of an **infinite** field of characteristic 7.

(b) [15 points] Give an example of a **finite** field with 49 elements.

Exercise 5 [20 minutes = 20 points] [`You cannot use here Theorem 17-6`]

Prove that the ideal $(X^2 + X + 1)$ is prime in $\frac{\mathbb{Z}}{2\mathbb{Z}}[X]$ but **not** in $\frac{\mathbb{Z}}{3\mathbb{Z}}[X]$.

Exercise 6 [20 minutes = 20 points]

Let $\phi: \mathbb{R} \to \mathbb{R}$ be an **onto** ring homomorphism. Prove:

(a) ϕ is a bijection (hence an isomorphism).

(b) $\phi(n) = n;\ \forall\, n \in \mathbb{Z}$.

(c) $\phi(q) = q;\ \forall\, q \in \mathbb{Q}$.

(d) $x > y \Rightarrow \phi(x) > \phi(y),\ \forall\, x, y \in \mathbb{R}$.

(e) ϕ is the identity.

Exercise 7 [20 minutes = 20 points]

Solve $x^{27} = 1$ in $\frac{\mathbb{Z}}{41\mathbb{Z}}$.

Exercise 8 [20 minutes = 20 points]

Find the remainder when X^{45} is divided by $X + 9$ in $\frac{\mathbb{Z}}{11\mathbb{Z}}[X]$.

Exercise 9 [20 minutes = 20 points]

Prove $\frac{\mathbb{Q}[X]}{(X^2-2)}$ is ring-isomorphic to $\mathbb{Q}[\sqrt{2}] := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.