Math 345 – Modern Algebra I (Spring 2015)

Major Exam 2 (Duration = 100 minutes)

Student Name_____ ID: _____

Exercise 1 [20 minutes = 20 points]

Determine all group homomorphisms from $\dfrac{\mathbb{Z}}{20\mathbb{Z}}$ to $\dfrac{\mathbb{Z}}{70\mathbb{Z}}$

Exercise 2 [20 minutes = 20 points]

Let $G := \{1,7,17,23,49,55,65,71\}$ be a group under multiplication modulo 96.

(a) Compute the orders of 7, 17, and 49.

(b) Determine the isomorphism class of G.

(c) Give 2 subgroups H and K of G such that G equals the **internal** direct product of H and K.

Exercise 3 [20 minutes = 20 points]

(a) Let R be a commutative ring and let $nil(R)$ denote the set of all nilpotent elements of R. Prove $nil(R)$ is an ideal of R; and it is included in every prime ideal of R.

(b) Let $R := \dfrac{\mathbb{Z}}{p^n\mathbb{Z}}$ with n a positive integer and p a prime integer. Determine $nil(R)$ and show it is a maximal ideal of R.

Exercise 4 [20 minutes = 20 points] [Justify your answers]

(a) [05 points] Give an example of an **infinite** integral domain of characteristic 3.

(b) [10 points] Construct an example of a ring R isomorphic to $\dfrac{\mathbb{Z}}{2\mathbb{Z}} \times \dfrac{\mathbb{Z}}{2\mathbb{Z}}$ as groups but **not** as rings.

(c) [05 points] Let $R := \dfrac{\mathbb{Z}}{5\mathbb{Z}}[i] := \left\{a + ib \mid a, b \in \dfrac{\mathbb{Z}}{5\mathbb{Z}}\right\}$. Is R a **field**?

Exercise 5 [20 minutes = 20 points]

Prove $\dfrac{\mathbb{R}[X]}{(X^2+X+1)}$ is a field.