King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

MATH 302 – Engineering Mathematics

Exam I (Term 142)

Duration: **120** Minutes March 04, 2015

Name : Solution key **ID #** :______________________

Section # : ___________ **Serial #:** __________

- Provide all **necessary steps** with **clear writing**.
- **Mobiles** and **calculators** are **NOT allowed** in the exam.

Question #	Marks	Maximum Marks
(1)		13
(2)		25
(3)		11
(4)		10
(5)		17
(6)		24
Total		**100**

(1)

(a) Define an ***orthonormal*** set of vectors.

Answer: A set of vectors is orthonormal if every pair of distinct vectors is orthogonal and each vector in the set is a unit vector.

(b) What is an ***orthogonal*** matrix?

Answer: An $n \times n$ nonsingular matrix A is orthogonal if $A^{-1} = A^T$.

(c) Suppose the system $\mathbf{AX} = \mathbf{B}$ is consistent, $\mathbf{A}$ is a 5×8 matrix and $\text{rank}(\mathbf{A}|\mathbf{B}) = 5$. How many ***parameters*** does the solution of the system have?

Answer: The solution of the system has $8 - 5 = 3$ parameters.

(d) Consider the homogeneous system with n equations in n variables $\mathbf{AX} = \mathbf{0}$. What can we say about the solution(s) of the system if

(i) $\mathbf{A}$ is singular?

Answer: The system has the nontrivial solutions (infinitely many solutions).

(ii) $\mathbf{A}$ is nonsingular?

Answer: The system has only the trivial solution (a unique solution).

(2) (a) Let λ be a real number and **A** be an $n \times n$ matrix. ***Show*** that $\boldsymbol{E}_\lambda = \{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}\}$ is a ***subspace*** of $\mathbb{R}^n$. ($\boldsymbol{E}_\lambda$ is called the eigenspace of **A** corresponding to the eigenvalue λ.)

Solution: $\boldsymbol{E}_\lambda$ is nonempty since $\mathbf{0}$ is in $\boldsymbol{E}_\lambda$.

Let $\boldsymbol{x}_1, \boldsymbol{x}_2$ be in $\boldsymbol{E}_\lambda$. We have $\mathbf{A}\,\boldsymbol{x}_1 = \lambda\,\boldsymbol{x}_1$ and $\mathbf{A}\,\boldsymbol{x}_2 = \lambda\,\boldsymbol{x}_2$.

Since $\mathbf{A}\,(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \mathbf{A}\,\boldsymbol{x}_1 + \mathbf{A}\,\boldsymbol{x}_2 = \lambda\,\boldsymbol{x}_1 + \lambda\,\boldsymbol{x}_2 = \lambda\,(\boldsymbol{x}_1 + \boldsymbol{x}_2)$, $\boldsymbol{x}_1 + \boldsymbol{x}_2$ is also in $\boldsymbol{E}_\lambda$.

Let k be a scalar. Since $\mathbf{A}\,(k\boldsymbol{x}_1) = k\mathbf{A}\,\boldsymbol{x}_1 = k\lambda\,\boldsymbol{x}_1 = \lambda\,(k\boldsymbol{x}_1)$, $k\boldsymbol{x}_1$ is also in $\boldsymbol{E}_\lambda$.

Hence, $\boldsymbol{E}_\lambda$ **is** a subspace of $\mathbb{R}^n$.

(b) Consider the vector space

$$\mathbf{S} = \{< u, v, x, y > \mid u + v + x = 0, y + 2u = v \text{ and } u, v, x, y \in \mathbb{R}\}.$$

Find a ***basis*** and the ***dimension*** of **S**.

Solution: Let $< u, v, x, y >$ be in **S**. We can write

$$\begin{aligned} < u, v, x, y > &= < u, v, -u - v, v - 2u > \\ &= < u, 0, -u, -2u > + < 0, v, -v, v > \\ &= u < 1,0,-1,-2 > + v < 0,1,-1,1 >. \end{aligned}$$

So, every vector in S is a linear combination of the vectors $\boldsymbol{a} =< 1,0,-1,-2 >$ and $\boldsymbol{b} =< 0,1,-1,1 >$.

Since $\boldsymbol{a}$ is not a scalar multiple of $\boldsymbol{b}$, the set $\mathbf{B} = \{\boldsymbol{a}, \boldsymbol{b}\}$ is linearly independent.

Hence, $\mathbf{B} = \{< 1,0,-1,-2 >, < 0,1,-1,1 >\}$ is a basis for **S.**

The dimension of **S** is 2.

(3) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 & -1 \\ 1 & -1 & 4 & 3 \\ 1 & -1 & 5 & 7 \end{pmatrix}.$$

(a) Find the ***rank*** of **A**.

Solution: Applying Gaussian Elimination to matrix **A** gives

$$\begin{pmatrix} 1 & -1 & 3 & -1 \\ 1 & -1 & 4 & 3 \\ 1 & -1 & 5 & 7 \end{pmatrix} \underset{-R_1+R_2}{\Longrightarrow} \begin{pmatrix} 1 & -1 & 3 & -1 \\ 0 & 0 & 1 & 4 \\ 1 & -1 & 5 & 7 \end{pmatrix}$$

$$\underset{-R_1+R_3}{\Longrightarrow} \begin{pmatrix} 1 & -1 & 3 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{pmatrix} \underset{-2R_2+R_3}{\Longrightarrow} \begin{pmatrix} 1 & -1 & 3 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, $\operatorname{rank}(\mathbf{A}) = \mathbf{2}$

(b) Is the set of row vectors of **A** linearly ***independent***? ***Justify*** your answer.

Answer: No, it is not. The set of row vectors of **A** is ***linearly dependent*** since rank(**A**) is less than the number of row vectors of **A.**

(4) Solve the following system using ***Gauss-Jordan Elimination*** method:

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ 2x_1 + 3x_2 + 5x_3 &= 8 \\ 4x_1 + 5x_3 &= 2 \end{aligned}$$

Solution: Applying ***Gauss-Jordan Elimination*** to the augmented matrix of the system gives

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right) \underset{-4R_1+R_3}{\overset{-2R_1+R_2}{\Longrightarrow}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right)$$

$$\underset{4R_2+R_3}{\overset{-R_2+R_1}{\Longrightarrow}} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 7 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right) \quad \overset{\frac{1}{13}R_3}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 7 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$$\underset{-3R_3+R_2}{\overset{2R_3+R_1}{\Longrightarrow}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right).$$

Hence, the solution of the system is $(x_1, x_2, x_3) = (3, 4, -2)$.

(5) (**a**) Find matrix $(\mathbf{AB})^{-1}$ if $\mathbf{A}=\begin{pmatrix}2 & -1 & 2\\ 1 & -1 & 2\\ -3 & 2 & -3\end{pmatrix}$ and $\mathbf{B}^{-1}=\begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}$.

Solution: Applying Gauss-Jordan Elimination to the matrix $(\mathbf{A}|\mathbf{I})$ gives

$$\left(\begin{array}{ccc|ccc}2 & -1 & 2 & 1 & 0 & 0\\ 1 & -1 & 2 & 0 & 1 & 0\\ -3 & 2 & -3 & 0 & 0 & 1\end{array}\right)\overset{R_{12}}{\Longrightarrow}\left(\begin{array}{ccc|ccc}1 & -1 & 2 & 0 & 1 & 0\\ 2 & -1 & 2 & 1 & 0 & 0\\ -3 & 2 & -3 & 0 & 0 & 1\end{array}\right)$$

$$\begin{matrix}-2R_1+R_2\\ \Longrightarrow\\ 3R_1+R_3\end{matrix}\left(\begin{array}{ccc|ccc}1 & -1 & 2 & 0 & 1 & 0\\ 0 & 1 & -2 & 1 & -2 & 0\\ 0 & -1 & 3 & 0 & 3 & 1\end{array}\right)\quad\begin{matrix}R_2+R_1\\ \Longrightarrow\\ R_2+R_3\end{matrix}\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & -1 & 0\\ 0 & 1 & -2 & 1 & -2 & 0\\ 0 & 0 & 1 & 1 & 1 & 1\end{array}\right)$$

$$\overset{2R_3+R_2}{\Longrightarrow}\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & -1 & 0\\ 0 & 1 & 0 & 3 & 0 & 2\\ 0 & 0 & 1 & 1 & 1 & 1\end{array}\right).$$

We obtain $\mathbf{A}^{-1}=\begin{pmatrix}1 & -1 & 0\\ 3 & 0 & 2\\ 1 & 1 & 1\end{pmatrix}$

and

$$(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}=\begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\ 3 & 0 & 2\\ 1 & 1 & 1\end{pmatrix}=\begin{pmatrix}4 & -1 & 2\\ 2 & 0 & 1\\ 4 & 1 & 3\end{pmatrix}$$

(**b**) Solve the system $(\mathbf{AB})\,\mathbf{X}=\mathbf{C}$, where **A and B** are the matrices given in (a),

$\mathbf{X}=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}$.

Solution:

$$\mathbf{X}=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=(\mathbf{AB})^{-1}\mathbf{C}=\begin{pmatrix}4 & -1 & 2\\ 2 & 0 & 1\\ 4 & 1 & 3\end{pmatrix}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}=\begin{pmatrix}6\\ 3\\ 7\end{pmatrix}.$$

(6) Let $\mathbf{A} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$.

(a) Explain why **A** is ***orthogonally diagonalizable***. **Answer:** Since **A** is a symmetric matrix.

(b) Find ***eigenvalues*** of **A.**

Solution: Characteristic equation of **A** is

$$\begin{vmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 4) - 2(-2\lambda - 4) + 2(4 + 2\lambda) = -(\lambda + 2)^2(\lambda - 4) = 0.$$

This gives the eigenvalues of **A:** $\lambda_1 = \lambda_2 = -2$ and $\lambda_3 = 4$.

(c) Find an ***orthogonal matrix*** **P** that diagonalizes **A** and find the ***diagonal matrix***

$\mathbf{D} = \mathbf{P}^T\mathbf{A}\mathbf{P}$.

Solution: For $\lambda_1 = \lambda_2 = -2$, we have

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right) \quad \textit{Row operation} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Two corresponding orthogonal eigenvectors are $K_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $K_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

For $\lambda_3 = 4$, we have

$$\left(\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right) \quad \textit{Row operation} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The corresponding eigenvector is $K_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

The required orthogonal matrix is $P = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$

and the diagonal matrix is $D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.