King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 301 Final Exam
The ~~First~~ 2^{ND} Semester of 2014-2015 (142)
Time Allowed: 180 Minutes

Name: SOLUTION ID#: ____________

Instructor: ____________ Sec #: ________ Serial #: ________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

Question #	Marks	Maximum Marks
1		12
2		22
3		22
4		22
5		22
6		20
7		20
Total		140

Q:1(8+4 points) Consider the Sturm–Liouville problem

$$y'' - \tan x y' + \lambda y = 0 \text{ with } y(\tfrac{\pi}{4}) = 0,\ y(\tfrac{\pi}{3}) = 0.$$

(a) Put the differential equation in self–adjoint form and write its weight function.

(b) If y_n and y_m are two eigenfunctions corresponding to two different eigenvalues, write the orthogonality relation.

(a) $IF = e^{\int -\tan x\, dx} = e^{Ln(\cos x)} = \cos x$ 2

$\cos x\, y'' - \tan x \cos x\, y' + \cos x\, \lambda y = 0$

$\cos x\, y'' - \sin x\, y' + \lambda \cos x\, y = 0$ 2

$\frac{d}{dx}[\cos x\, y'] + \lambda \cos x\, y = 0$ 2

weight function $p(x) = \cos x$ 2

(b) Orthogonality relation

$$\int_{\pi/4}^{\pi/3} y_n(x)\, y_m(x)\, p(x)\, dx = 0 \quad \text{for } n \neq m$$

3 1

Q:2 (22 points) Use separation of variables method to solve the problem

$$5\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi, \ t > 0,$$

subject to the boundary and initial conditions

$$\begin{aligned} u_x(0,t) &= 0, \quad u_x(\pi,t) = 0, \quad t > 0, \\ u(x,0) &= x, \quad 0 < x < \pi. \end{aligned}$$

Let $u(x,t) = X(x)\,T(t)$, (1)

$$X''T = \frac{XT'}{5}$$

$$\frac{X''}{X} = \frac{T'}{5T} = -\lambda$$

$X'' + \lambda X = 0$, $\quad T' + 5\lambda T = 0$ (2)

$u_x(0,t) = 0 \Rightarrow X'(0) = 0$

$u_x(\pi,t) = 0 \Rightarrow X'(\pi) = 0$

Case I $\lambda = 0$. $X(x) = c_1 + c_2 x$

$X'(x) = c_2$

$X'(0) = 0 \Rightarrow c_2 = 0$

$X'(\pi) = 0 \Rightarrow c_2 = 0$

$X(x) = c_1 \neq 0$

Non trivial solution (02)

Case II $\lambda = -\alpha^2 < 0$, $\alpha > 0$

$X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x$

$X'(x) = c_1 \alpha \sinh \alpha x + c_2 \alpha \cosh \alpha x$

$X'(0) = 0 \Rightarrow c_2 = 0$

$X'(\pi) = 0 \Rightarrow c_1 \underbrace{\alpha \sinh \alpha \pi}_{\neq 0} = 0$

$\Rightarrow c_1 = 0$

Trivial solution. (02)

Case III $\lambda = \alpha^2 > 0$, $\alpha > 0$ (03)

$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$

$X'(x) = -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x$

$X'(0) = 0 \Rightarrow c_2 = 0$

$X'(\pi) = 0 \Rightarrow -c_1 \alpha \sin \alpha \pi = 0$

Let $c_1 \neq 0$, $\sin \alpha \pi = 0 \Rightarrow \alpha \pi = n\pi$

$\Rightarrow \alpha = n$, $n = 1, 2, \ldots$

$X(x) = c_1 \cos nx$

Now for $\lambda = 0$, $T(t) = c_3$ constant

For $\lambda = n^2$, $T(t) = c_5 e^{-5n^2 t}$ (2)

$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-5n^2 t} \cos nx$ (2)

$u(x,0) = x = A_0 + \sum_{n=1}^{\infty} A_n \cos nx$ (2)

$A_0 = \frac{1}{\pi}\int_0^\pi x\, dx = \frac{\pi}{2}$ (01)

$A_n = \frac{2}{\pi}\int_0^\pi x \cos nx\, dx$

$= \frac{2}{\pi}\left[x \frac{\sin nx}{n} \Big|_0^\pi - \int_0^\pi \frac{\sin nx}{n}\, dx \right]$

$= \frac{2}{n^2 \pi} \cos nx \Big|_0^\pi = \frac{2((-1)^n - 1)}{n^2 \pi}$ (03)

$u(x,t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{n^2 \pi} e^{-5n^2 t} \cos nx$

(02)

Q:3 (22 points) Use Laplace transform to solve the problem

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad x > 0,\ t > 0,$$

subject to the boundary and initial conditions

$$\begin{aligned} u(0,t) &= f(t), \quad \lim_{x\to\infty} u(x,t) = 0, \quad t > 0, \\ u(x,0) &= 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < 1, \end{aligned}$$

where $f(t) = \begin{cases} \sin \pi t & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$

$$4\frac{d^2U(x,s)}{dx^2} = s^2U(x,s) - sU(x,0) - U_t(x,0) \quad 2 + 2$$

$$\frac{d^2U}{dx^2} - \frac{s^2}{4}U = 0 \quad 2$$

$$U(x,s) = C_1 e^{-\frac{s}{2}x} + C_2 e^{\frac{s}{2}x} \quad 2$$

$$\lim_{x\to\infty} u(x,t) = 0 \overset{\mathcal{L}}{\Rightarrow} \lim_{x\to\infty} U(x,s) = 0$$

$$\overset{\text{set}}{\Rightarrow} \quad C_2 = 0 \quad 2$$

$$U(x,s) = C_1 e^{-\frac{s}{2}x} \quad 2$$

$$u(0,t) = [1 - \mathcal{U}(t-1)]\sin \pi t$$

$$= \sin \pi t - \sin \pi t\, \mathcal{U}(t-1)$$

$$U(0,s) = \frac{\pi}{s^2+\pi^2} - e^{-s}\mathcal{L}\{\sin \pi(t+1)\}$$

$$= \frac{\pi}{s^2+\pi^2} - e^{-s}\mathcal{L}\{\sin(\pi t + \pi)\}$$

$$= \frac{\pi}{s^2+\pi^2} - e^{-s}\mathcal{L}\{-\sin \pi t\}$$

$$U(0,s) = \frac{\pi}{s^2+\pi^2} + \frac{e^{-s}\pi}{s^2+\pi^2} \quad 04$$

$$\Rightarrow C_1 = \frac{\pi}{s^2+\pi^2} + \frac{\pi}{s^2+\pi^2}e^{-s} \quad 02$$

$$U(x,s) = \frac{\pi}{s^2+\pi^2}e^{-\frac{x}{2}s} + \frac{\pi}{s^2+\pi^2}e^{-(\frac{x}{2}+1)s}$$

$$u(x,t) = \sin \pi\left(t - \tfrac{x}{2}\right)\mathcal{U}\left(t - \tfrac{x}{2}\right) + \sin \pi\left(t - \tfrac{x}{2} - 1\right)\mathcal{U}\left(t - \tfrac{x}{2} - 1\right) \quad 4$$

Q:4 (22 points) Use separation of variables method to solve the problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0,\ 0 < r < 2,\ 0 < z < 5,$$

subject to the boundary conditions

$$\begin{aligned} u(2,z) &= 0,\ 0 < z < 5 \\ u(r,0) &= 0,\ 0 < r < 2 \\ u(r,5) &= 4,\ 0 < r < 2 \end{aligned}$$

Also solution $u(r,z)$ is bounded at r =0.

Let $u(r,z) = R(r)\,Z(z)$, then

$$R''Z + \frac{1}{r}R'Z = -RZ''$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{Z''}{Z} = -\lambda \quad (02)$$

$$R'' + \frac{1}{r}R' + \lambda R = 0, \qquad Z'' - \lambda Z = 0$$

$$r^2R'' + rR' + \lambda r^2 R = 0 \quad (02)$$

Parametric Bessel equation with $n=0$, $\lambda = \alpha^2$.

(02) $R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$

$Y_0(\alpha r) \to -\infty$ as $r \to 0^+$

$\Rightarrow c_2 = 0$ (02)

$u(2,z) = 0 \Rightarrow R(2) = 0$

$\Rightarrow c_1 J_0(2\alpha) = 0$

Let $c_1 \neq 0$, $\alpha = \alpha_i$, $i = 1,2,3,\ldots$ are non zero values of α such that $J_0(2\alpha_i) = 0$, $i = 1,2,3,\ldots$

So $R(r) = c_1 J_0(\alpha_i r)$ (02)

For $\lambda = \alpha_i^2$, $Z'' - \alpha_i^2 Z = 0$

$\Rightarrow Z(z) = c_3 \cosh \alpha_i z + c_4 \sinh \alpha_i z$ (02)

$u(r,0) = 0 \Rightarrow Z(0) = 0$

$\Rightarrow c_3 = 0$

$Z(z) = c_3 \sinh \alpha_i z$ (02)

$$u(r,z) = \sum_{i=1}^{\infty} A_i \sinh \alpha_i z\, J_0(\alpha_i r)$$

$$u(r,5) = 4 = \sum_{i=1}^{\infty} A_i \sinh 5\alpha_i\, J_0(\alpha_i r) \quad (02)$$

$$\sinh 5\alpha_i\, A_i = \frac{2}{4 J_1^2(2\alpha_i)} \int_0^2 4\, r J_0(\alpha_i r)\,dr$$

$$A_i = \frac{2}{\sinh 5\alpha_i\, J_1^2(2\alpha_i)} \frac{1}{\alpha_i^2} \int_0^{2\alpha_i} t\, J_0(t)\,dt$$

$$= \frac{2}{\sinh 5\alpha_i\, J_1^2(2\alpha_i)\, \alpha_i^2} \cdot 2\alpha_i J_1(2\alpha_i)$$

$$= \frac{4}{\sinh 5\alpha_i\, J_1(2\alpha_i)\, \alpha_i} \quad (04)$$

$$u(r,z) = \sum_{i=1}^{\infty} \frac{4 \sinh \alpha_i z}{\alpha_i \sinh 5\alpha_i\, J_1(2\alpha_i)} J_0(\alpha_i r)$$

(02)

Q:5 (22 points) Find the steady-state temperature $u(r,\theta)$ in a sphere of radius 2 by solving the problem

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}=0,\quad 0<r<2,\ 0<\theta<\pi,$$

subject to the boundary condition

$$u(2,\theta) = 1+\cos(\theta),\quad 0<\theta<\pi.$$

Let $u(r,\theta)=R(r)\Theta(\theta)$, then

$$R''\Theta+\frac{2}{r}R'\Theta=-\frac{1}{r^2}R\Theta''-\frac{\cot\theta}{r^2}R\Theta'$$

$$\frac{r^2R''}{R}+\frac{2rR'}{R}=-\frac{\Theta''}{\Theta}-\frac{\cot\theta\,\Theta'}{\Theta}=\lambda$$

$$r^2R''+2rR'-\lambda R=0$$

$$\Theta''+\cot\theta\,\Theta'+\lambda\Theta=0 \quad 04$$

Let $x=\cos\theta$, then

$$\Theta'=\frac{\partial\Theta}{\partial\theta}=\frac{\partial\Theta}{\partial x}\frac{\partial x}{\partial\theta}=-\sin\theta\frac{\partial\Theta}{\partial x}$$

$$\Theta''=\frac{\partial^2\Theta}{\partial\theta}=-\cos\theta\frac{\partial\Theta}{\partial x}+\sin^2\theta\frac{\partial^2\Theta}{\partial x^2}$$

$$\Rightarrow \sin^2\theta\frac{\partial^2\Theta}{\partial x^2}-\cos\theta\frac{\partial\Theta}{\partial x}-\sin\theta\cot\theta\frac{\partial\Theta}{\partial x}+\lambda\Theta=0$$

$$(1-\cos^2\theta)\frac{\partial^2\Theta}{\partial x^2}-2\cos\theta\frac{\partial\Theta}{\partial x}+\lambda\Theta=0$$

$$03 \quad (1-x^2)\frac{\partial^2\Theta}{\partial x^2}-2x\frac{\partial\Theta}{\partial x}+\lambda\Theta=0$$

Legendre equation, $\lambda=n(n+1)$

$$\Theta_n(\theta)=P_n(\cos\theta)$$

$n=0,1,2,\dots$ 03

Now for $\lambda=n(n+1)$

$$r^2R''+2rR'-n(n+1)=0$$

$$m(m-1)+2m-n^2-n=0$$

$$m=-(n+1),\quad m=n$$

$$R(r)=C_1r^n+C_2r^{-(n+1)}$$

$R(r)\to\infty$ as $r\to 0 \Rightarrow C_2=0$

$$R(r)=C_1r^n \quad 04$$

$$u(r,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta) \quad 02$$

$$u(2,\theta)=1+\cos\theta=\sum_{n=0}^{\infty}A_n2^nP_n(\cos\theta)$$

$$\Rightarrow 2^nA_n=\frac{2n+1}{2}\int_0^{\pi}(1+\cos\theta)P_n(\cos\theta)\sin\theta\,d\theta$$

$$A_n=\frac{2n+1}{2^{n+1}}\left[\int_0^{\pi}P_0(\cos\theta)P_n(\cos\theta)\sin\theta\,d\theta+\int_0^{\pi}P_1(\cos\theta)P_n(\cos\theta)\sin\theta\,d\theta\right]$$

$A_n=0$ for $n\neq 0,1$

$$A_0=\frac{1}{2}\int_0^{\pi}\sin\theta\,d\theta=-\frac{1}{2}\cos\theta\Big|_0^{\pi}=1$$

$$A_1=\frac{3}{4}\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta$$

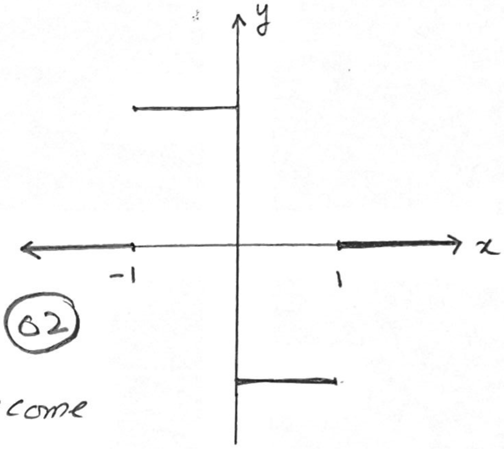
$$=-\frac{3}{4}\frac{\cos^3\theta}{3}\Big|_0^{\pi}=\frac{1}{2}$$

$$u(r,\theta)=1+\frac{1}{2}r\cos\theta$$

03

Q:6 (20 points) Find Fourier integral representation of

$$f(x) = \begin{cases} 0, & x < -1 \\ 2, & -1 < x < 0 \\ -2, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$



Since * f is an odd function, 02

the Fourier integral will become Fourier Sine integral.

$$f(x) = \frac{2}{\pi}\int_0^\infty B(\alpha)\, \sin\alpha x \, d\alpha \qquad 04$$

$$B(\alpha) = \int_0^\infty f(x)\, \sin\alpha x \, dx \qquad 04$$

$$= -\int_0^1 2\, \sin\alpha x \, dx \qquad 2 + 04$$

$$= \left.\frac{2\cos\alpha x}{\alpha}\right|_0^1 = \frac{2}{\alpha}(\cos\alpha - 1) \qquad 04$$

$$f(x) = \frac{4}{\pi}\int_0^\infty \frac{\cos\alpha - 1}{\alpha}\, \sin\alpha x \, d\alpha \qquad 02$$

* or 02 $A(\alpha) = 0$ (by calculation)

Q:7 (20 points) Use appropriate Fourier transform to find the temperature $u(x,t)$ in a semi-infinite rod if $u_x(0,t) = 0$ and $u(x,0) = e^{-x}$, $x > 0$.

Use Fourier Cosine transform,

$$K \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad (04)$$

$$\Rightarrow \quad -\alpha^2 U(\alpha,t) - u_x(0,t) = \frac{1}{K}\frac{\partial U}{\partial t} \qquad (04)$$

$$(02) \quad \frac{\partial U}{\partial t} + K\alpha^2 U = 0 \quad \Rightarrow U(\alpha,t) = c\, e^{-K\alpha^2 t} \qquad (02)$$

$$u(x,0) = e^{-x}$$

$$\Rightarrow U(\alpha,0) = \int_0^\infty e^{-x} \cos\alpha x \, dx$$

$$= e^{-x}\frac{\sin\alpha x}{\alpha}\Big|_0^\infty + \int_0^\infty e^{-x}\frac{\sin\alpha x}{\alpha}\,dx$$

$$= \frac{1}{\alpha}\left[-e^{-x}\frac{\cos\alpha x}{\alpha}\Big|_0^\infty - \int_0^\infty e^{-x}\frac{\cos\alpha x}{\alpha}\,dx\right]$$

$$= \frac{1}{\alpha^2} - \frac{1}{\alpha^2}U(\alpha,0) \qquad (04)$$

$$\frac{\alpha^2+1}{\alpha^2}U(\alpha,0) = \frac{1}{\alpha^2}$$

$$U(\alpha,0) = \frac{1}{1+\alpha^2} \quad \Rightarrow \quad C = \frac{1}{1+\alpha^2} \qquad (02)$$

$$U(\alpha,t) = \frac{1}{1+\alpha^2}e^{-K\alpha^2 t}$$

$$\Rightarrow u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\alpha^2}e^{-K\alpha^2 t}\cos\alpha x\,d\alpha \qquad (02)$$

Note: No point for solving wave or Laplace equation.