King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 301 Major Exam 2
The Second Semester of 2014-2015 (142)

Time Allowed: 120 Minutes

Name: SOLUTION ID#: ______

Instructor: ______ Sec #: ______ Serial #: ______

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

Question #	Marks	Maximum Marks
1		15
2		15
3		12
4		12
5		10
6		10
7		14
8		12
Total		100

Q:1 (7+8 points) Find the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s+1)^2}\right\}$, by using convolution,

(b) Write $f(t) = \begin{cases} \sin(t), & 0 \le t < \pi \\ e^{2t}\cos(t), & t \ge \pi \end{cases}$

in compact form using unit step function and find its Laplace transform.

(a) $\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{(s+1)^2}\right\} = \int_0^t \tau e^{-\tau}\, d\tau$ ③ ①

$= -(1+\tau)e^{-\tau}\Big|_0^t$ ②

$= -(1+t)e^{-t} + 1$ ①

(b) $f(t) = \sin t - \sin t\, \mathcal{U}(t-\pi) + e^{2t}\cos t\, \mathcal{U}(t-\pi)$ ③

$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} - e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\}$

$+ e^{-\pi s}\mathcal{L}\{e^{2(t+\pi)}\cos(t+\pi)\}$ ②

$= \frac{1}{s^2+1} + e^{-\pi s}\mathcal{L}\{\sin t\}$

$- e^{-\pi s}e^{2\pi}\mathcal{L}\{e^{2t}\cos t\}$ ②

$= \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} - \frac{s-2}{(s-2)^2+1}e^{\pi(2-s)}$ ①

Q:2 (5+5+5 points) Find the following:

(a) $\mathcal{L}\{te^{-2t}\cos 3t\}$,

(b) $\mathcal{L}\{f(t)\}$, where $f(t) = \begin{cases} 2t+1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$.

(c) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2s-3}\right\}$.

(a) $\mathcal{L}\{t\,e^{-2t}\cos 3t\} = -\dfrac{d}{ds}\left(\dfrac{s+2}{(s+2)^2+9}\right)$ (3)

$$= -\frac{(s+2)^2+9-2(s+2)^2}{((s+2)^2+9)^2}$$

$$= \frac{(s+2)^2-9}{((s+2)^2+9)^2}$$ (2)

(b) $f(t) = (2t+1) - (2t+1)\,\mathcal{U}(t-1)$ (1)

$\mathcal{L}\{f(t)\} = \dfrac{2}{s^2} + \dfrac{1}{s} - e^{-s}\mathcal{L}\{2(t+1)+1\}$ (2)

$= \dfrac{2}{s^2} + \dfrac{1}{s} - e^{-s}\left(\dfrac{2}{s^2} + \dfrac{3}{s}\right)$ (2)

(c) $\mathcal{L}^{-1}\left\{\dfrac{s+1-1}{(s+1)^2-4}\right\} = \mathcal{L}^{-1}\left\{\dfrac{s+1}{(s+1)^2-2^2}\right\} - \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{2}{(s+1)^2-2^2}\right\}$ (2) (2)

$= e^{-t}\cosh 2t - \dfrac{1}{2}e^{-t}\sinh 2t$ (1)

$\dfrac{s}{(s+3)(s-1)} = \dfrac{A}{s+3} + \dfrac{B}{s-1}$

$s = A(s-1) + B(s+3)$

$s = 1 \quad 1 = 4B \quad B = \frac{1}{4}$

$s = -3 \quad -3 = -4A \quad A = \frac{3}{4}$

$\mathcal{L}\left\{\dfrac{s}{s^2+2s-3}\right\} = \dfrac{3}{4}e^{-3t} + \dfrac{1}{4}e^{t}$

Q:3 (12 points) Solve the following boundary value problem using Laplace transform

$y'' + 9y = \cos 2t$ with $y(0) = 1$, $y(\frac{\pi}{2}) = -1$.

Let $y'(0) = c$ (2)

$$s^2 Y(s) - s y(0) - y'(0) + 9 Y(s) = \frac{s}{s^2+4}$$

$$(s^2+9) Y(s) = \frac{s}{(s^2+4)} + s + c$$

$$Y(s) = \frac{s}{(s^2+9)(s^2+4)} + \frac{s}{s^2+9} + \frac{c}{s^2+9} \quad (3)$$

$$\frac{s}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4}$$

$$s = A(s^3+4s) + B(s^2+4) + C(s^3+9s) + D(s^2+9)$$

Compare the coefficients of:

s^3	$0 = A + C$	$B = 0$	$D = 0$
s^2	$0 = B + D$	$C = \frac{1}{5}$	$A = -\frac{1}{5}$
s	$1 = 4A + 9C$		
s^0	$0 = 4B + 9D$		

$$Y(s) = -\frac{1}{5}\frac{s}{s^2+9} + \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{c}{s^2+9} + \frac{s}{s^2+9} \quad (3)$$

$$y(t) = -\frac{1}{5}\cos 3t + \frac{1}{5}\cos 2t + \frac{c}{3}\sin 3t + \cos 3t \quad (2)$$

$$-1 = y(\tfrac{\pi}{2}) = -\frac{1}{5}\cancel{\cos\frac{3\pi}{2}} + \frac{1}{5}\cos 2\frac{\pi}{2} + \frac{c}{3}\sin\frac{3\pi}{2}$$
$$+ \cancel{\cos\frac{3\pi}{2}}$$

$$-1 = -\frac{1}{5} - \frac{c}{3}$$

$$\frac{c}{3} = 1 - \frac{1}{5} = \frac{4}{5} \qquad c = \frac{12}{5} \quad (2)$$

$$y(t) = \frac{4}{5}\cos 3t + \frac{1}{5}\cos 2t + \frac{4}{5}\sin 3t$$

Q:4 (12 points) Solve the following Volterra integral equation $f(t)+\frac{8}{3}\int\limits_0^t f(\tau)(\tau-t)^3 d\tau = 1+t$.

$$f(t) - \frac{8}{3}\int_0^t f(\tau)\,(t-\tau)^3\,d\tau = 1+t \qquad 2$$

$$F(s) - \frac{8}{3}F(s)\cdot\frac{3!}{s^4} = \frac{1}{s}+\frac{1}{s^2}$$

$$F(s)\left[1-\frac{16}{s^4}\right] = \frac{s+1}{s^2}$$

$$F(s) = \frac{s^2(s+1)}{s^4-16} = \frac{s^3+s^2}{(s-2)(s+2)(s^2+4)} \qquad 3$$

$$\frac{s^3+s^2}{(s-2)(s+2)(s^2+4)} = \frac{A}{s-2}+\frac{B}{s+2}+\frac{Cs+D}{s^2+4}$$

$$s^3+s^2 = A(s+2)(s^2+4)+B(s-2)(s^2+4)$$
$$+\,Cs(s^2-4)+D(s^2-4)$$

Put $s=2$ $\quad 8+4=32A \quad A=\frac{12}{32}=\frac{3}{8}$

$s=-2$ $\quad -8+4=-32B \quad B=\frac{-4}{-32}=\frac{1}{8}$

Compare the coefficients of:

s^3 $\quad 1=A+B+C \quad C=1-\frac{1}{2}=\frac{1}{2}$

s^2 $\quad 1=2A-2B+D \quad D=1-\frac{1}{2}=\frac{1}{2}$

$$F(s) = \frac{3/8}{s-2}+\frac{1/8}{s+2}+\frac{1}{2}\,\frac{s}{s^2+4}+\frac{1}{4}\,\frac{2}{s^2+4} \qquad 4$$

$$f(t) = \frac{3}{8}e^{2t}+\frac{1}{8}e^{-2t}+\frac{1}{2}\cos 2t+\frac{1}{4}\sin 2t \qquad 3$$

Q:5 (10 points) Solve the following initial value problem using Laplace transform

$y'' + 6y' + 10y = \delta(t - 2\pi)$ with $y(0) = 1$, $y'(0) = 1$.

$$s^2 Y(s) - s y(0) - y'(0) + 6s Y(s) - 6y(0) + 10 Y(s) = e^{-2\pi s}$$ 3

$$(s^2 + 6s + 10) Y(s) = e^{-2\pi s} + s + 7$$ 2

$$Y(s) = \frac{e^{-2\pi s}}{(s+3)^2 + 1} + \frac{s+3}{(s+3)^2+1} + \frac{4}{(s+3)^2+1}$$ 2

$$y(t) = e^{-3(t-2\pi)} \sin(t-2\pi)\, \mathcal{U}(t-2\pi)$$

$$+ e^{-3t} \cos t + 4 e^{-3t} \sin t$$ 3

Q:6 (10 points) Show that $f_1(x) = x^3$ and $f_2(x) = x^2 + 1$ are orthogonal on $[-1, 1]$. Find values of a and b such that both $f_1(x)$ and $f_2(x)$ are orthogonal to $f_3(x) = ax + bx^2 + x^3$.

$$(f_1, f_2) = \int_{-1}^{1} (x^5 + x^3)\, dx = \frac{x^6}{6} + \frac{x^4}{4} \Big|_{-1}^{1} = 0 \qquad 2$$

$$(f_1, f_3) = \int_{-1}^{1} (ax^4 + bx^5 + x^6)\, dx$$

$$= \frac{ax^5}{5} + \frac{bx^6}{6} + \frac{x^7}{7} \Big|_{-1}^{1}$$

$$= \frac{2a}{5} + 0 + \frac{2}{7} = 0 \qquad a = -\frac{5}{7} \qquad 4$$

$$(f_2, f_3) = \int_{-1}^{1} [ax^3 + bx^4 + x^5 + ax + bx^2 + x^3]\, dx$$

$$= (a+1)\frac{x^4}{4} + \frac{bx^5}{5} + \frac{x^6}{6} + \frac{ax^2}{2} + \frac{bx^3}{3} \Big|_{-1}^{1}$$

$$= 0 + \frac{2b}{5} + 0 + 0 + \frac{2b}{3} = 0$$

$$b = 0 \qquad 4$$

Q:7 (14 points) Find the Fourier series of the function $f(x) = \begin{cases} 0 & -\frac{\pi}{2} < x < 0 \\ \cos x & 0 \le x < \frac{\pi}{2} \end{cases}$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2nx + b_n \sin 2nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x \, dx = \frac{2}{\pi} \sin x \Big|_0^{\frac{\pi}{2}} = \frac{2}{\pi} \quad 2$$

$$a_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x \cos 2nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} [\cos(2n+1)x + \cos(2n-1)x] \, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(2n+1)x}{2n+1} + \frac{\sin(2n-1)x}{2n-1}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right] = \frac{(-1)^n}{\pi}\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right]$$

$$= \frac{2(-1)^{n+1}}{\pi(4n^2-1)} \quad 5$$

$$b_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin 2nx \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} [\sin(2n+1)x + \sin(2n-1)x] \, dx$$

$$= \frac{-1}{\pi}\left[\frac{\cos(2n+1)x}{2n+1} + \frac{\cos(2n-1)x}{2n-1}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left[\frac{1}{2n+1} + \frac{1}{2n-1}\right] = \frac{4n}{\pi(4n^2-1)} \quad 5$$

$$f(x) = \frac{1}{\pi} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi(4n^2-1)} \cos 2nx + \frac{4n}{\pi(4n^2-1)} \sin 2nx$$

2

Q:8 (8+4 points) **(a)** Find the half-range Fourier cosine expansion of $f(x) = \sin 4x, \quad 0 \le x < \frac{\pi}{8}$.

(b) Use part **(a)** to show that $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$.

(a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 8nx$ ①

$a_0 = \frac{16}{\pi} \int_0^{\frac{\pi}{8}} \sin 4x \, dx = \frac{-16}{\pi} \frac{\cos 4x}{4} \Big|_0^{\frac{\pi}{8}}$

$= -\frac{4}{\pi}(0 - 1) = \frac{4}{\pi}$ ②

$a_n = \frac{16}{\pi} \int_0^{\frac{\pi}{8}} \sin 4x \cos 8nx \, dx$

$= \frac{8}{\pi} \int_0^{\frac{\pi}{8}} [\sin (4+8n)x + \sin(4-8n)x] \, dx$

$= \frac{-8}{\pi} \left[\frac{\cos(4+8n)x}{4+8n} + \frac{\cos(4-8n)x}{4-8n} \right]_0^{\frac{\pi}{8}}$

$= \frac{8}{\pi} \left[\frac{1}{4+8n} + \frac{1}{4-8n} \right] = \frac{8}{\pi} \frac{8}{16 - 64n^2}$

$= \frac{4}{\pi} \frac{1}{1 - 4n^2}$ ④

$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)} \cos 8nx$ ①

(b) Put $x = 0$ ②

$0 = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)}$

$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$ ②