King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 201 - Term 142 - Final Exam

Duration: 180 minutes

KEY

Name:____________________ ID Number:____________________

Section Number:____________________ Serial Number:____________________

ClassTime: ____________________ Instructor's Name:____________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 10 pages of problems (Total of 10 Problems)

Page Number	Points	Maximum Points
1		18
2		10
3		14
4		14
5		13
6		11
7		14
8		16
9		14
10		16
Total		140

1. (a) (9-points) Find an equation of the plane through the point $P(2,1,-1)$ and containing the line $L: x=t,\ y=-t+2,\ z=4t+3$.

direction vector of L is $\vec{d} = \langle 1,-1,4 \rangle$ (1pt)

a point on L is $S(0,2,3)$ (1pt)

the vector $\vec{SP} = \langle 2,-1,-4 \rangle$ (1pt)

the vector $\vec{n} = \vec{SP} \times \vec{d}$ is normal to the plane (2pts)

$$\vec{n} = \underbrace{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -4 \\ 1 & -1 & 4 \end{vmatrix}}_{\text{(1pt)}} = \underbrace{\langle -8,-12,-1 \rangle}_{\text{(1pt)}}$$

Using point S, an equation of the plane is

$$-8x - 12(y-2) - (z-3) = 0 \quad \text{(2pts)}$$

or

$$8x + 12y + z = 25$$

(b) (9-points) Determine whether the lines given by the parametric equations

$$L_1: x = 1+2t,\ y = 3t,\ z = 3-t; \ -\infty < t < \infty,$$

$$L_2: x = -1+s,\ y = 4+s,\ z = 2-3s; \ -\infty < s < \infty$$

are parallel, intersecting, or skew (neither parallel nor intersecting).

direction vector of L_1 is $\vec{d_1} = \langle 2,3,-1 \rangle$ (1pt)

direction vector of L_2 is $\vec{d_2} = \langle 1,1,-3 \rangle$ (1pt)

Since $\vec{d_1}$ and $\vec{d_2}$ are not linearly dependant, L_1 and L_2 are not parallel (1pt)

check whether they intersect.

$$\underbrace{\begin{cases} 1+2t = -1+s \\ 3t = 4+s \end{cases}}_{\text{(1pt)}} \Rightarrow \begin{cases} 2t-s = -2 \\ 3t-s = 4 \end{cases} \Rightarrow \begin{cases} 2t-s = -2 \\ t = 6 \end{cases} \Rightarrow \underbrace{\begin{matrix} s = 14 \\ t = 6. \end{matrix}}_{\text{(1pt)}}$$

when $t=6$, z-coordinate of the point on L_1 is -3 (1pt)

when $s=14$, z-coordinate of the point on L_2 is -50 (1pt)

This means L_1 and L_2 are not intersecting (1pt)

Hence, they are skew. (1pt)

2. (10-points) Identify the symmetries of the curve $r^2 = 4\sin\theta$ and sketch it.

Assume (r, θ) is on the curve.

- symmetry about the y-axis:

Since $\sin(\pi - \theta) = \sin(\theta)$, the point $(r, \pi - \theta)$ is on the curve. 1pt

Then the curve is symmetric about the y-axis. 1pt

- symmetry about the origin:

Since $(-r)^2 = r^2$, the point $(-r, \theta)$ is on the curve. 1pt

Then the curve is symmetric about the origin. 1pt

- symmetry about the x-axis:

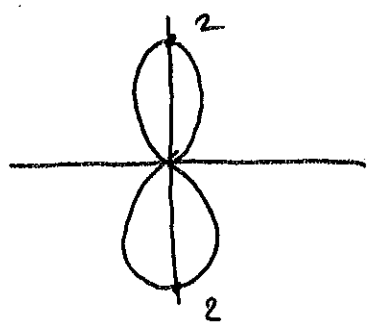
Since $(-r)^2 = r$ and $\sin(\pi - \theta) = \sin\theta$, that is since the curve is already symmetric about the y-axis and about the origin, the point $(-r, \pi - \theta)$ is on the curve. 1pt

Then the curve is symmetric about the x-axis. 1pt

- sketching the curve

θ	r
0	0
$\pi/2$	± 2
π	0

2 pts*



2 pts

*should have at least 3 correct points for full mark.

3. (a) (7-points) Determine whether the limit of $f(x,y) = \dfrac{x^4y}{x^2+y^2}$ exists or not as $(x,y) \to (0,0)$. Justify your answer.

In polar coordinates, $f(x,y) = \dfrac{x^4y}{x^2+y^2}$ is $f(r,\theta) = r^3\sin\theta\cos\theta$. (2pts)

For any (r,θ), $-r^3 \leq r^3\sin\theta\cos\theta \leq r^3$ (2 pts)

Since $\underbrace{\lim_{r\to 0} r^3 = 0}_{\text{1 pt}}$, by Sandwich Theorem

$\lim_{r\to 0} r^3\sin\theta\cos\theta = 0$. (1 pt)

Then $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. (1 pt)

(b) (7-points) Find the linearization $L(x,y,z)$ of $f(x,y,z) = 2xze^{zy^2}$ at $(3,1,-2)$.

$f_x(x,y,z) = 2ze^{zy^2}$ (1pt) $\rightsquigarrow$ $f_x(3,1,-2) = -4e^{-2}$

$f_y(x,y,z) = 4xyz^2e^{zy^2}$ (1pt) $\rightsquigarrow$ $f_y(3,1,-2) = 48e^{-2}$

$f_z(x,y,z) = (2x+2xy^2z)e^{zy^2}$ (1pt) $\rightsquigarrow$ $f_z(3,1,-2) = -6e^{-2}$

(1 pt)

$f(3,1,-2) = -12e^{-2}$ (1 pt)

$L(x,y,z) = f(3,1,-2) + f_x(3,1,-2)(x-3) + f_y(3,1,-2)(y-1) + f_z(3,1,-2)(z+2)$ (1pt)

$L(x,y,z) = (-4x + 48y - 6z - 60)e^{-2}$ (1pt)

4. (a) (5-points) If $f(x,y,z) = x^2 \sin\left(\frac{y}{z}\right)$, then find $\frac{\partial^3 f}{\partial x \partial y \partial z}$ at $(1, \pi/3, 2)$.

$f_x(x,y,z) = 2x\sin\left(\frac{y}{z}\right)$ (1 pt)

$f_{xy}(x,y,z) = 2\frac{x}{z}\cos\left(\frac{y}{z}\right)$ (1 pt)

$f_{xyz}(x,y,z) = 2x\left[-\sin\left(\frac{y}{z}\right)\cdot\frac{-y}{z^2}\cdot\frac{1}{z} + \cos\left(\frac{y}{z}\right)\cdot\frac{-1}{z^2}\right]$ (1 pt)

$= 2x\left[\frac{y}{z^3}\sin\left(\frac{y}{z}\right) - \frac{1}{z^2}\cos\left(\frac{y}{z}\right)\right]$

$f_{xyz}(1, \pi/3, 2) = 2\left[\frac{\pi/3}{8}\sin\left(\frac{\pi/3}{2}\right) - \frac{1}{4}\cos\left(\frac{\pi/3}{2}\right)\right]$ (1 pt)

$= \frac{\pi}{24} - \frac{\sqrt{3}}{4}$ (1 pt)

(b) (9-points) Find the points on the ellipsoid $4x^2 + y^2 + 4z^2 = 14$ where $f(x,y,z) = 2x - 2y + 6z$ has its maximum and minimum values.

We use the method of Lagrange multipliers.

Let $g(x,y,z) = 4x^2 + y^2 + 4z^2$

$\nabla f(x,y,z) = \langle 2, -2, 6 \rangle$ (1 pt)

$\nabla g(x,y,z) = \langle 8x, 2y, 8z \rangle$ (1 pt)

$$\underbrace{\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ 4x^2 + y^2 + 4z^2 = 14 \end{cases}}_{\text{2 pts}} \Rightarrow \begin{cases} 2 = 8\lambda x \\ -2 = 2\lambda y \\ 6 = 8\lambda z \\ 4x^2 + y^2 + 4z^2 = 14 \end{cases}$$ (1 pt)

From the first three equations in the system, $\lambda \neq 0$. Then

$$\begin{cases} x = \frac{1}{4\lambda} \\ y = \frac{-1}{\lambda} \\ z = \frac{3}{4\lambda}. \\ 4x^2 + y^2 + 4z^2 = 14 \end{cases} \Rightarrow \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{9}{4\lambda^2} = 14 \Rightarrow \lambda = \pm\frac{1}{2}$$

We find two points $\left(\frac{1}{2}, -2, \frac{3}{2}\right)$ and $\left(-\frac{1}{2}, 2, -\frac{3}{2}\right)$ (1 pt) (1 pt)

$f\left(\frac{1}{2}, -2, \frac{3}{2}\right) = 14$ and $f\left(-\frac{1}{2}, 2, -\frac{3}{2}\right) = -14$.

So f has max at $\left(\frac{1}{2}, -2, \frac{3}{2}\right)$ (1 pt), and min at $\left(-\frac{1}{2}, 2, -\frac{3}{2}\right)$ (1 pt)

5. (13-points) Find all local maxima, local minima, and saddle points of

$$f(x, y) = 6xy - x^3 + 3y^2.$$

$f_x(x,y) = 6y - 3x^2$ (1pt) $f_y(x,y) = 6x + 6y$ (1pt)

Critical Points:

$$\underbrace{\begin{matrix} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{matrix}}_{(1pt)} \Rightarrow \begin{cases} 6y - 3x^2 = 0 \\ 6x + 6y = 0 \end{cases} \Rightarrow \begin{cases} x = -y \\ 3x^2 + 6x = 0 \end{cases} \Rightarrow$$

$(0,0)$ (1pt) and $(-2,2)$ (1pt) are critical points.

2nd derivative test to classify the critical points:

$$\underbrace{\left.\begin{matrix} f_{xx}(x,y) = -6x \\ f_{yy}(x,y) = 6 \\ f_{xy}(x,y) = 6 \end{matrix}\right\}}_{(1pt)} \Rightarrow \underbrace{\Delta(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - f_{xy}^2(x,y)}_{(1pt)} = \underbrace{-36x - 36}_{(1pt)}$$

at $(0,0)$:

$\underbrace{\Delta(0,0) = -36 < 0}_{(1pt)}$, $\underbrace{\text{then } (0,0) \text{ is a saddle point.}}_{(1pt)}$

at $(-2,2)$:

$\underbrace{\Delta(-2,2) = 36 > 0}_{(1pt)}$ and $\underbrace{f_{xx}(-2,2) = 12 > 0}_{(1pt)}$, then

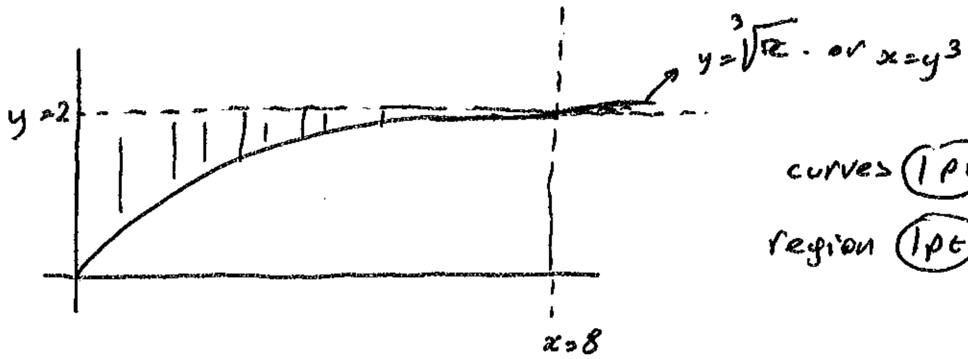
$\underbrace{(-2,2) \text{ is a local minimum.}}_{(1pt)}$

6. (11-points) Sketch the region of integration for the double integral $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dydx}{y^8+1}$, reverse the order of integration, and evaluate the integral.

The region of integration is:

$R = \{(x,y) \mid 0 \leq x \leq 8,\ \sqrt[3]{x} \leq y \leq 2\}$ (1pt)

Its sketch is:



curves (1pt)

region (1pt)

Upon reversing the order,

$$R = \{(x,y) \mid \underbrace{0 \leq y \leq 2}_{\text{(1pt)}},\ \underbrace{0 \leq x \leq y^3}_{\text{(1pt)}}\}$$

Then,

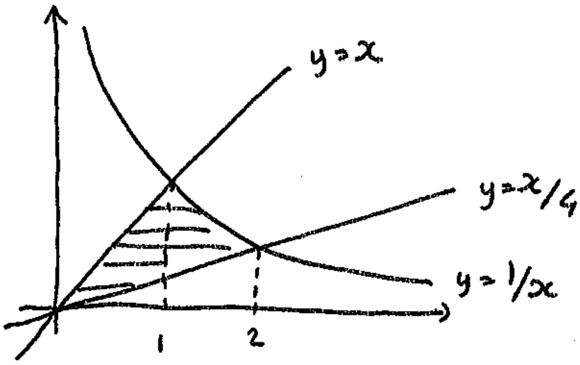
$$\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^8+1}\,dy\,dx = \underbrace{\int_0^2 \int_0^{y^3} \frac{1}{y^8+1}\,dx\,dy}_{\text{(1pt)}} = \int_0^2 \frac{x}{y^8+1}\Big|_0^{y^3} dy = \underbrace{\int_0^2 \frac{y^3}{y^8+1}\,dy}_{\text{(1pt)}}$$

Using the substitution, $\begin{cases} u = y^4 & du = 4y^3 \\ y=0 & \Rightarrow u=0 \\ y=2 & \Rightarrow u=16 \end{cases}$ (1pt)

$$= \underbrace{\frac{1}{4}\int_0^{16} \frac{du}{u^2+1}}_{\text{(1pt)}} = \underbrace{\frac{1}{4}\tan^{-1}(u)\Big|_0^{16}}_{\text{(1pt)}} = \underbrace{\frac{1}{4}\tan^{-1}(16)}_{\text{(1pt)}}.$$

7. (14-points) Find the volume of the solid whose base is the region in the first quadrant of the xy-plane enclosed by the curves $y = x$, $y = x/4$, and $y = 1/x$ while the top of the solid is bounded by the plane $z = x + 4$.

First we sketch the region.



curves → 1 pt

intersections → 1 pt

region → 1 pt

$$\text{Volume} = \int_0^1 \int_{x/4}^{x} \underbrace{x+4 \, dy\, dx}_{1pt} + \int_1^2 \int_{x/4}^{1/x} \underbrace{x+4 \, dy\, dx}_{1pt}$$

1pt 1pt 1pt 1pt

$$= \int_0^1 xy + 4y \Big|_{x/4}^{x} dx + \int_1^2 xy + 4y \Big|_{x/4}^{1/x} dx. \quad (1+1 = 2\text{ pts})$$

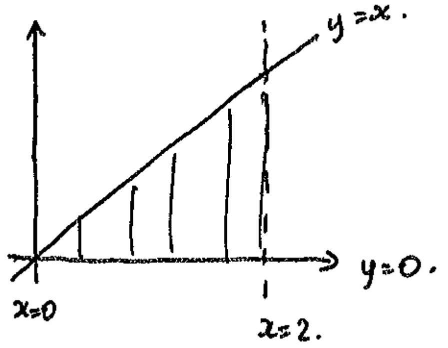
$$= \int_0^1 \frac{3x^2}{4} + 3x \, dx + \int_1^2 -\frac{x^2}{4} - x + 1 + \frac{4}{x} \, dx$$

$$= \frac{x^3}{4} + \frac{3}{2}x^2 \Big|_0^1 + \left(-\frac{x^3}{12} - \frac{x^2}{2} + x + 4\ln x\right)\Big|_1^2 \quad (1+1 = 2\text{ pts})$$

$$= \frac{2}{3} + 4\ln 2 \quad 1\text{ pt}$$

8. Consider the region $R = \{(x,y)|0 \le x \le 2,\ 0 \le y \le x\}$.

(a) (8-points) Sketch the region R, find its area and describe it in polar coordinates.



The region R is a right triangle and its area is $\frac{2\times 2}{2} = 2$ (1pt)

curves → (1pt)

region → (1pt)

(1pt) $\begin{cases} x=0 \Rightarrow r=0. \\ x=2 \Rightarrow r=2\sec\theta \end{cases}$

(1pt) $\begin{cases} y=0 \Rightarrow \theta=0. \\ y=x \Rightarrow \theta=\pi/4. \end{cases}$

$R = \{(r,\theta) \mid \underbrace{0 \le r \le 2\sec\theta}_{1pt},\ \underbrace{0 \le \theta \le \pi/4}_{1pt}\}.$ (1pt)

(b) (8-points) Find the average value of $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$ over the region R.

In polar coordinates $\dfrac{1}{\sqrt{x^2+y^2}}$ is $\dfrac{1}{r}$ (1pt)

$$\text{Avg} = \underbrace{\frac{1}{\text{Area of } R}}_{1pt} \int_{\underbrace{0}_{1pt}}^{\pi/4} \int_{\underbrace{0}_{1pt}}^{2\sec\theta} \underbrace{\frac{1}{r}\, r\, dr\, d\theta}_{1pt}.$$

$$= \frac{1}{2} \int_0^{\pi/4} \int_0^{2\sec\theta} dr\, d\theta = \underbrace{\int_0^{\pi/4} \sec\theta\, d\theta}_{1pt}$$

$$= \underbrace{\ln|\sec\theta + \tan\theta| \Big|_0^{\pi/4}}_{1pt} = \underbrace{\ln(\sqrt{2}+1)}_{1pt}$$

9. (14-points) Use **cylindrical coordinates** to find the volume of the solid enclosed by the surfaces $z = x^2 + y^2$ and $z = 6 - x^2 - y^2$.

In cylindrical coordinates,

- the surface $z = x^2 + y^2$ is $z = r^2$ (1 pt)
- the surface $z = 6 - x^2 - y^2$ is $z = 6 - r^2$ (1 pt)
- the intersection of the surface is

$r^2 = 6 - r^2 \Rightarrow r^2 = 3 \Rightarrow r = \sqrt{3}$ (1 pt)

- the description of the solid D is

$$D = \{(r, \theta, z) \mid \underbrace{0 \le \theta \le 2\pi}_{1\,pt}, \underbrace{0 \le r \le \sqrt{3}}_{1\,pt}, \underbrace{r^2 \le z \le 6 - r^2}_{1\,pt}\}$$

$$\text{Volume of } D = \underset{1\,pt}{\int_0^{2\pi}} \underset{1\,pt}{\int_0^{\sqrt{3}}} \underset{1\,pt}{\int_{r^2}^{6-r^2}} \underbrace{r\, dz\, dr\, d\theta}_{1\,pt} = \underbrace{2\pi \int_0^{\sqrt{3}} \int_{r^2}^{6-r^2} r\, dz\, dr}_{1\,pt}$$

$$= 2\pi \int_0^{\sqrt{3}} r z \Big|_{r^2}^{6-r^2} dr = \underbrace{2\pi \int_0^{\sqrt{3}} r(6 - 2r^2)\, dr}_{1\,pt}$$

$$= 2\pi \int_0^{\sqrt{3}} 6r - 2r^3\, dr = \underbrace{2\pi \left(3r^2 - \frac{r^4}{2}\right)\Big|_0^{\sqrt{3}}}_{1\,pt}$$

$$= 9\pi \quad (1\,pt)$$

10. (16-points) Use **spherical coordinates** to find the volume of the solid inside the sphere $x^2 + y^2 + (z-1)^2 = 1$ and outside the cone $z = \sqrt{3x^2 + 3y^2}$.

In spherical coordinates,

- the sphere $x^2+y^2+(z-1)^2=1$ has the equation

$$x^2+y^2+z^2=2z \Rightarrow \rho^2 = 2\rho\cos\phi \Rightarrow \rho = 2\cos\phi \quad (1pt)$$

- the cone $z=\sqrt{3x^2+3y^2}$ has the equation

$$\rho\cos\phi = \sqrt{3\rho^2\sin^2\phi} \Rightarrow \underbrace{\tan\phi = \frac{1}{\sqrt{3}}}_{(1pt)} \Rightarrow \underbrace{\phi = \pi/6.}_{(1pt)}$$

- the description of the solid D is

$$D = \{(\rho,\phi,\theta) \mid \underbrace{0\le\rho\le 2\cos\phi}_{(1pt)}, \underbrace{\pi/6 \le \phi \le \pi/2}_{(1pt)}, \underbrace{0\le\theta\le 2\pi}_{(1pt)}\}$$

$$\text{Volume of } D = \underset{(1pt)}{\int_0^{2\pi}} \underset{(1pt)}{\int_{\pi/6}^{\pi/2}} \underset{(1pt)}{\int_0^{2\cos\phi}} \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{(1pt)}$$

$$= 2\pi \int_{\pi/6}^{\pi/2}\int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \quad (1pt)$$

$$= \frac{2\pi}{3}\int_{\pi/6}^{\pi/2} \rho^3 \sin\phi \Big|_0^{2\cos\phi} d\phi$$

$$= \frac{16\pi}{3}\int_{\pi/6}^{\pi/2} \cos^3\phi \sin\phi \, d\phi \quad (1pt)$$

using substitution

$u=\cos\phi$ $du=-\sin\phi$

$\phi=\pi/6 \Rightarrow u=\sqrt{3}/2$

$\underbrace{\phi=\pi/2 \Rightarrow u=0}_{(1pt)}$

$$= \underbrace{-\frac{16\pi}{3}\int_{\sqrt{3}/2}^{0} u^3 \, du}_{(1pt)} = \underbrace{-\frac{4\pi}{3} u^4 \Big|_{\sqrt{3}/2}^{0}}_{(1pt)} = \underbrace{\frac{3\pi}{4}}_{(1pt)},$$