King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 201 - Term 142 - Exam I

Duration: 120 minutes

Name:_______________ ID Number:_______________

Section Number:_______________ Serial Number:_______________

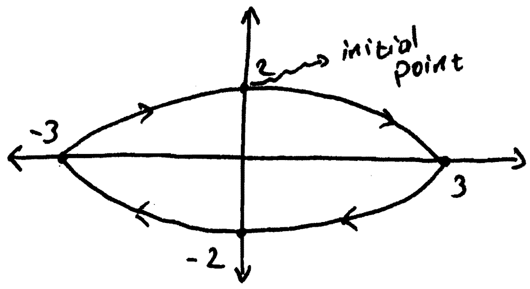
ClassTime: _______________ Instructor's Name:_______________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 8 pages of problems (Total of 8 Problems)

Question Number	Points	Maximum Points
1		15
2		15
3		13
4		12
5		10
6		9
7		14
8		12
Total		100

1. a) (7-points) Find parametric equations and an interval of time for the motion of a particle that starts at the point $(0, 2)$ tracing the ellipse $4x^2 + 9y^2 = 36$ twice in clockwise direction.



t	x	y
0	0	2
$\pi/2$	3	0
π	0	-2
$3\pi/2$	-3	0
2π	0	2

(1 pt)

$x = 3\cos(\pi/2 - t) = 3\sin t$ (2 pts)

$y = 2\sin(\pi/2 - t) = 2\cos t$ (2 pts)

The table shows that to trace the ellipse once in clockwise direction an interval of time of $[0, 2\pi]$ is needed. Therefore to trace it twice an interval of time of $[0, 4\pi]$ is needed. (2 pts)

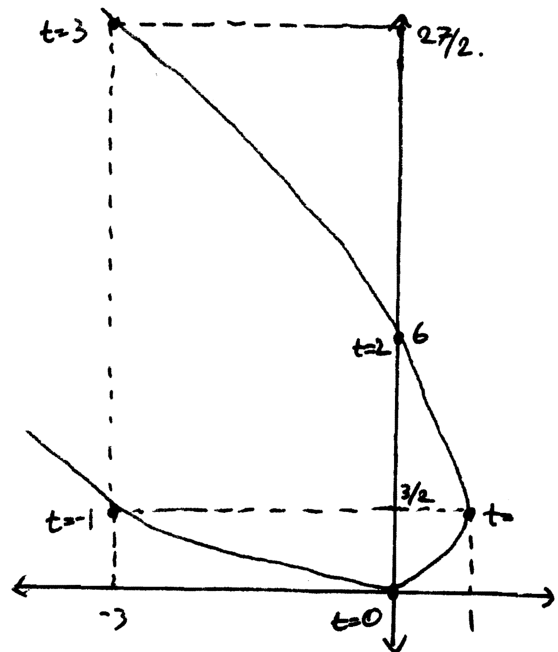
Note: There are other possible solutions.

b) (8-points) Find the area enclosed by the parametric curve $x = 2t - t^2$, $y = \frac{3}{2}t^2$ and the y-axis.

t	x	y
-1	-3	$3/2$
0	0	0
1	1	$3/2$
2	0	6
3	-3	$27/2$

(2* pts)

* If a student finds y-int without a table he should be awarded the 2 points allocated for the table.



$$\text{Area} = \underbrace{\int_0^6 x\,dy}_{(3\text{ pts})} = \underbrace{\int_0^2 (2t - t^2)\,3t\,dt}_{(1\text{ pt})} = 3\int_0^2 2t^2 - t^3\,dt$$

$$= \underbrace{3\left(\frac{2}{3}t^3 - \frac{t^4}{4}\right)\Big|_0^2}_{(1\text{ pt})} = \underbrace{4}_{(1\text{ pt})}$$

2. Let C be a curve given by the parametric equations $x = \frac{1}{t}$, $y = t - \sin(\pi t)$, $t > 0$.

a) (10-points) Find the equation of the tangent line to the curve C at $t = 1$.

Slope of the tangent line at $t=1$ is $\left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right|_{t=1}$ (2 pts)

$$\left.\begin{aligned} \frac{dy}{dt} &= 1 - \pi\cos(\pi t) \quad \text{(1 pt)} \\ \frac{dx}{dt} &= -1/t^2 \quad \text{(1 pt)} \end{aligned}\right\} \Rightarrow \left.\frac{dy}{dx}\right|_{t=1} = \underbrace{\frac{1-\pi\cos(\pi)}{-1}}_{\text{(1 pt)}} = \underbrace{-1-\pi}_{\text{(1 pt)}}$$

the point corresponding to the parameter $t=1$ is $(1, 1)$ (2 pts)

equation of the tangent line: $y - 1 = (-1-\pi)(x-1)$ (2 pts)

b) (5-points) Find $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \quad \text{(2 pts)}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{1-\pi\cos(\pi t)}{-1/t^2}\right) = 2\pi t\cos(\pi t) - \pi^2 t^2 \sin(\pi t) - 2t \quad \text{(1 pts)}$$

Then $$\frac{d^2y}{dx^2} = \frac{2\pi t\cos(\pi t) - \pi^2 t^2\sin(\pi t) - 2t}{-1/t^2}$$

$$= \pi^2 t^4 \sin(\pi t) + 2t^3 - 2\pi t^3\cos(\pi t) \quad \text{(1 pt)}$$

3. Let C be a curve given by the polar equation $r = \sec\theta(\tan\theta + 2)$.

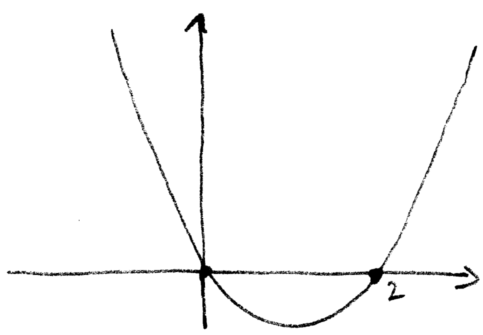
(a) (4-points) Find an equivalent cartesian equation for the curve C.

$$r = \sec\theta(\tan\theta + 2) \Rightarrow r\cos\theta = \tan\theta + 2 \quad (1\ pt)$$

$$\Rightarrow x = \frac{y}{x} + 2 \quad (2\ pts)$$

$$\Rightarrow x^2 - 2x = y \quad (1\ pt)$$

(b) (3-points) Sketch and identify the graph of C.



- C is a parabola (1 pt)
- sketch (1 pt)
- x - intercepts (1 pt)

(c) (6-points) Find the slope of the tangent line to the curve C at $\theta = \pi/4$.

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \left.\frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}\right|_{\theta=\pi/4} \quad (2\ pts)$$

$$\frac{dr}{d\theta} = \sec\theta\tan\theta(\tan\theta+2) + \sec^3\theta \quad (1\ pt)$$

$$\left.\frac{dr}{d\theta}\right|_{\theta=\pi/4} = 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2} \quad (1\ pt)$$

$$r(\pi/4) = 3\sqrt{2} \quad (1\ pt)$$

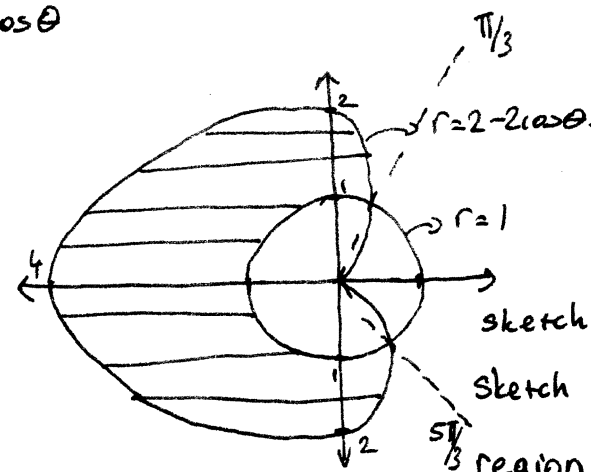
Then $$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{5\sqrt{2}\cdot\frac{1}{\sqrt{2}} + 3\sqrt{2}\cdot\frac{1}{\sqrt{2}}}{5\sqrt{2}\cdot\frac{1}{\sqrt{2}} - 3\sqrt{2}\cdot\frac{1}{\sqrt{2}}} = 4 \quad (1\ pt)$$

<u>Note</u>: If a student solves part c) using cartesian eqn of the curve and the point in cart. coord. that corresponds to $\theta = \pi/4$ then he should be awarded the 7 points allocated for part c).

4. (12-points) Find the area of the region outside the circle $r = 1$ and inside the cardioid $r = 2 - 2\cos\theta$.

table for $r=2-2\cos\theta$

θ	r
0	0
$\pi/2$	2
π	4
$3\pi/2$	2
2π	0



sketch of $r=1$ (1pt)
sketch of $r=2-2\cos\theta$ (1pt)
region (1pt)
* if a student sketches $r=2-2\cos\theta$ correctly without the table he should be awarded the 1 point.

intersection points:

$2-2\cos\theta=1 \Rightarrow \cos\theta=\frac{1}{2} \Rightarrow \theta=\pi/3, 5\pi/3$ (2 pts)

$$\text{Area} = \frac{1}{2}\int_{\pi/3}^{5\pi/3} (2-2\cos\theta)^2 - 1^2 \, d\theta \quad (3 \text{ pts})$$

$$= \frac{1}{2}\int_{\pi/3}^{5\pi/3} 3 - 8\cos\theta + 4\cos^2\theta \, d\theta$$

$$= \frac{1}{2}\int_{\pi/3}^{5\pi/3} 3 - 8\cos\theta + 4\left(\frac{\cos 2\theta + 1}{2}\right) d\theta \quad (1 \text{ pt})$$

$$= \int_{\pi/3}^{5\pi/3} \frac{5}{2} - 4\cos\theta + 2\cos 2\theta \, d\theta$$

$$= \frac{5}{2}\theta - 4\sin\theta + \frac{1}{2}\sin 2\theta \Big|_{\pi/3}^{5\pi/3} \quad (1 \text{ pt})$$

$$= \frac{10}{3}\pi - 4\left(\sin\frac{5\pi}{3} - \sin\frac{\pi}{3}\right) + \frac{1}{2}\left(\sin\frac{10}{3}\pi - \sin\frac{2\pi}{3}\right) \quad (1 \text{ pt})$$

$$= \frac{10}{3}\pi + \frac{7}{2}\sqrt{3} \quad (1 \text{ pt})$$

5. Let S be a sphere whose one of diameters has end points at $A(1,4,-2)$ and $B(-7,1,2)$.

(a) (6-points) Find an equation of S.

The center C of S is the midpoint of A and B (1 pt)

Then $C = \left(\frac{1-7}{2}, \frac{4+1}{2}, \frac{-2+2}{2}\right) = \left(-3, \frac{5}{2}, 0\right)$ (1 pt)

The radius r of S is the half of the distance between A and B (1 pt)

Then $r = \frac{1}{2}\sqrt{(-7-1)^2+(1-4)^2+(2+2)^2} = \frac{1}{2}\sqrt{89}$ (1 pt)

Equation of S : $(x+3)^2 + \left(y-\frac{5}{2}\right)^2 + z^2 = \frac{89}{4}$ (2 pts)

(b) (4-points) Describe the intersection of S and the xz-plane.

The intersection is ;

$$\underbrace{(x+3)^2 + \left(0-\frac{5}{2}\right)^2 + z^2 = \frac{89}{4}}_{1\text{ pt}} \Rightarrow \underbrace{(x+3)^2+z^2 = 16}_{1\text{ pt}}$$

This is a circle on the xz-plane with center $(-3,0,0)$ and radius 4. (2 pts)

6. (9-points) Find the point in the xy-plane that is equidistant from the points $(1, 3, \sqrt{2})$ and $(2, 4, -4)$ and has a y-coordinate equal to three times its x-coordinate.

Let $P(x, y, z)$ be the point that we want to find.

- P is in the xy-plane $\Rightarrow z = 0$ 1 pt
- y-coord. is three times the x-coord. $\Rightarrow y = 3x$ 1 pt
- P is equidistant from A and B $\Rightarrow |AP| = |BP|$ 1 pt

$|AP|^2 = (x-1)^2 + (y-3)^2 + (z-\sqrt{2})^2$ 1 pt

$|BP|^2 = (x-2)^2 + (y-4)^2 + (z-4)^2$ 1 pt

Then we need to solve the system:

$$\begin{cases} (x-1)^2 + (y-3)^2 + (z-\sqrt{2})^2 = (x-2)^2 + (y-4)^2 + (z-4)^2 \\ y = 3x \\ z = 0 \end{cases}$$ 1 pt

$\Rightarrow (x-1)^2 + (3x-3)^2 + 2 = (x-2)^2 + (3x-4)^2 + 16 \Rightarrow x = 3$ 1 pt

$\Rightarrow y = 9$ 1 pt

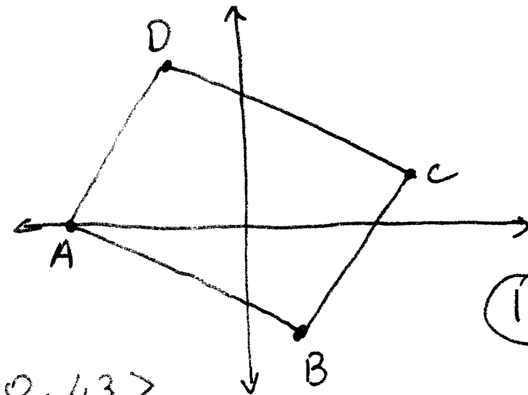
The point P is $(3, 9, 0)$. 1 pt

7. a) (6-points) Find the area of the parallelogram whose vertices are $A(-6,0)$, $B(1,-4)$, $C(3,1)$, and $D(-4,5)$.

$\vec{AB} = \langle 7, -4 \rangle$ (1 pt)

$\vec{AD} = \langle 2, 5 \rangle$ (1 pt)

Area $= |\vec{AB} \times \vec{AD}|$ (1 pt)



(1 pt)

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} i & j & k \\ 7 & -4 & 0 \\ 2 & 5 & 0 \end{vmatrix} = \langle 0, 0, 43 \rangle$$ (1 pt)

Then area is 43. (1 pt)

b) (8-points) Use dot product to decide whether the triangle with vertices $P(-1,2,3)$, $Q(2,-2,0)$, and $R(3,1,-4)$ is right-angled.

$\underbrace{\vec{PQ} = \langle 3, -4, -3 \rangle}_{(1\text{ pt})}$, $\underbrace{\vec{PR} = \langle 4, -1, -7 \rangle}_{(1\text{ pt})}$, $\underbrace{\vec{QR} = \langle 1, 3, -4 \rangle}_{(1\text{ pt})}$

$\vec{PQ} \cdot \vec{PR} = 12 + 4 + 21 \neq 0 \Rightarrow \vec{PQ} \not\perp \vec{PR}$ (1 pt)

$\vec{PQ} \cdot \vec{QR} = 3 - 12 + 12 \neq 0 \Rightarrow \vec{PQ} \not\perp \vec{QR}$ (1 pt)

$\vec{PR} \cdot \vec{QR} = 4 - 3 + 28 \neq 0 \Rightarrow \vec{PR} \not\perp \vec{QR}$ (1 pt)

So none of the angles are right angles. (1 pt)

The triangle is not a right triangle. (1 pt)

8. (12-points) Let $\vec{a} = < 1, -2, -2 >$ and $\vec{b} = < 1, 0, -1 >$. Find the angle between the vectors $\vec{a}$ and $\text{proj}_{\vec{b}}\vec{a} - \vec{a}$.

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} \quad \text{1 pt}$$

$\vec{a}\cdot\vec{b} = 3$ 1 pt $\qquad |\vec{b}| = \sqrt{2}$ 1 pt

$$\text{proj}_{\vec{b}}\vec{a} = \frac{3}{2}<1,0,-1> = <\frac{3}{2}, 0, -\frac{3}{2}> \quad \text{1 pt}$$

$$\text{proj}_{\vec{b}}\vec{a} - \vec{a} = <\frac{3}{2}, 0, -\frac{3}{2}> - <1,-2,-2> = <\frac{1}{2}, 2, \frac{1}{2}> \quad \text{1 pt}$$

If θ is the angle between $\vec{a}$ and $\text{proj}_{\vec{b}}\vec{a}$, then

$$\cos\theta = \frac{\vec{a}\cdot(\text{proj}_{\vec{b}}\vec{a} - \vec{a})}{|\vec{a}|\,|\text{proj}_{\vec{b}}\vec{a} - \vec{a}|} \quad \text{2 pts}$$

$|\vec{a}| = 3$ 1 pt $\qquad |\text{proj}_{\vec{b}}\vec{a} - \vec{a}| = \frac{3}{\sqrt{2}}$ 1 pt

$$\vec{a}\cdot(\text{proj}_{\vec{b}}\vec{a} - \vec{a}) = <1,-2,-2>\cdot<\frac{1}{2}, 2, \frac{1}{2}> = -\frac{9}{2} \quad \text{1 pt}$$

Then $\cos\theta = \dfrac{-9/2}{9/\sqrt{2}} = \underbrace{-\dfrac{\sqrt{2}}{2}}_{\text{1 pt}} \Rightarrow \underbrace{\theta = 3\pi/4}_{\text{1 pt}}$