King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 102 - Exam II - Term 141

Duration: 90 minutes

KEY

Name:________________ ID Number:________________

Section Number:________________ Serial Number:________________

Class Time: ________________ Instructor's Name:________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 6 pages of problems (Total of 8 Problems)

Question Number	Points	Maximum Points
1		24
2		10
3		10
4		12
5		10
6		10
7		10
8		14
Total		100

1. Evaluate the following integrals:

a) (**8 points**) $\int \frac{\log_3(x^2)}{x}\,dx$

Let $u = \log_3(x^2)$. Then $du = \frac{1}{(\ln 3)\, x^2} \cdot 2x\, dx = \frac{2}{\ln 3} \cdot \frac{1}{x}\, dx$ ② ②

$$\int \frac{\log_3(x^2)}{x}\,dx = \frac{\ln 3}{2}\int u\,du \quad ②$$

$$= \frac{\ln 3}{2} \cdot \frac{1}{2} u^2 + C \quad ①$$

$$= \frac{1}{4}\ln 3\,\left[\log_3(x^2)\right]^2 + C \quad ①$$

b) (**8 points**) $\int \cos^2 x\ \sin^3 x\,dx$

$$= \int \cos^2 x \cdot \sin^2 x \cdot \sin x\,dx \quad ①$$

$$= \int \cos^2 x \cdot (1 - \cos^2 x) \cdot \sin x\,dx \quad ②$$

$$= \int (\cos^2 x - \cos^4 x) \cdot \sin x\,dx$$

$u = \cos x \implies du = -\sin x\,dx$ ②

$$= -\int (u^2 - u^4)\,du \quad ①$$

$$= -\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C \quad ① \quad = -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C \quad ①$$

c)(**8 points**) $\int \sec\theta\ \csc\theta\ d\theta$

$$= \int \frac{1}{\cos\theta\,\sin\theta}\,d\theta \quad ②$$

$$= \int \frac{2}{\sin(2\theta)}\,d\theta \quad ②$$

$$= \int 2\csc(2\theta)\,d\theta \quad ②$$

$$= -\ln\left|\csc(2\theta) + \cot(2\theta)\right| + C \quad ②$$

$$\overset{\text{or}}{=} \ln\left|\csc(2\theta) - \cot(2\theta)\right| + C$$

2. (**10 points**) If $\coth^2 x = \dfrac{25}{16}$, $x < 0$, then find the value of $\sinh(2x)$.

$\bullet\ \coth^2 x - 1 = \operatorname{csch}^2 x$ (1)

$\Longrightarrow \operatorname{csch}^2 x = \dfrac{25}{16} - 1 = \dfrac{9}{16}$ (1)

$\Longrightarrow \sinh^2 x = \dfrac{16}{9}$ (1) $\Longrightarrow \sinh x = -\dfrac{4}{3}$, as $x < 0$. (1)

$\bullet\ \cosh^2 x - \sinh^2 x = 1$ (1) $\Longrightarrow \cosh^2 x = 1 + \dfrac{16}{9} = \dfrac{25}{9}$ (1)

$\Longrightarrow \cosh x = \dfrac{5}{3}$ (1) (as $\cosh x > 0$ for all x)

$\bullet\ \sinh(2x) = 2 \sinh x \cosh x$ (2)

$= 2 \cdot \dfrac{-4}{3} \cdot \dfrac{5}{3} = -\dfrac{40}{9}$ (1)

3. (**10 points**) Determine whether the sequence $\{a_n\}_{n=1}^{\infty}$ converges or diverges. If it converges, find its limit

$$a_n = \sqrt{4n^2 - 3} - \sqrt{4n^2 - n}$$

$\bullet\ \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\sqrt{4n^2-3} - \sqrt{4n^2-n}\right) \cdot \dfrac{\sqrt{4n^2-3} + \sqrt{4n^2-n}}{\sqrt{4n^2-3} + \sqrt{4n^2-n}}$ (2)

$= \lim\limits_{n\to\infty} \dfrac{-3+n}{\sqrt{4n^2-3} + \sqrt{4n^2-n}}$ (2)

$= \lim\limits_{n\to\infty} \dfrac{-3+n}{n\left[\sqrt{4-\frac{3}{n^2}} + \sqrt{4-\frac{1}{n}}\right]}$

$= \lim\limits_{n\to\infty} \dfrac{n\left(-\frac{3}{n}+1\right)}{n\left(\sqrt{4-\frac{3}{n^2}} + \sqrt{4-\frac{1}{n}}\right)}$

$= \lim\limits_{n\to\infty} \dfrac{-\frac{3}{n}+1}{\sqrt{4-\frac{3}{n^2}} + \sqrt{4-\frac{1}{n}}}$ (2)

$= \dfrac{0+1}{\sqrt{4}+\sqrt{4}} = \dfrac{1}{4}$ (2)

$\bullet$ The sequence <u>Converges</u> and its limit is $\frac{1}{4}$.

(2)

4. (**12 points**) Evaluate $\int \frac{3x^3+4x^2+2}{x^4+x^2}\,dx$ $= I$

$\boxed{1+1+2}$

- $\frac{3x^3+4x^2+2}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$

$\Longrightarrow 3x^3+4x^2+2 = A\,x(x^2+1) + B(x^2+1) + (Cx+D)x^2$

$\boxed{\text{1 pt for each constant}}$

- $x=0 \Longrightarrow \boxed{2=B}$
- Coeff. of x : $\boxed{0=A}$
- Coff of x^2 : $4 = B + D \Longrightarrow \boxed{D=2}$
- Coff of x^3 : $3 = A + C \Longrightarrow \boxed{C=3}$

- $I = \int \frac{2}{x^2} + \frac{3x+2}{x^2+1}\,dx$ (1)

$= \int \frac{2}{x^2} + \frac{3x}{x^2+1} + \frac{2}{x^2+1}\,dx$ $\boxed{1+1+1}$

$= -\frac{2}{x} + \frac{3}{2}\ln(x^2+1) + 2\tan^{-1}x + C$

5. (**10 points**) Evaluate the integral $\int \frac{x^5}{\sqrt{4-x^4}}\,dx$ by starting first with the substitution $t = x^2$.

Let $t = x^2$. Then $dt = 2x\,dx$ 1

$$I = \int \frac{x^5}{\sqrt{4-x^4}}\,dx = \int \frac{x^4}{\sqrt{4-x^4}} \cdot x\,dx = \frac{1}{2}\int \frac{t^2}{\sqrt{4-t^2}}\,dt \quad 1$$

Let $t = 2\sin\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then 1

$dt = 2\cos\theta\,d\theta$ 1

& $\sqrt{4-t^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$, as $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. 1

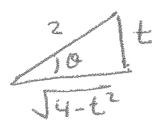
$$I = \frac{1}{2}\int \frac{4\sin^2\theta}{2\cos\theta} \cdot 2\cos\theta\,d\theta$$

$$= 2\int \sin^2\theta\,d\theta \quad 1$$

$$= \int 1 - \cos(2\theta)\,d\theta \quad 1$$

$$= \theta - \frac{1}{2}\sin(2\theta) + C \quad 1$$

$$= \theta - \sin\theta\cos\theta + C$$



$$= \sin^{-1}\left(\frac{t}{2}\right) - \frac{t}{2} \cdot \frac{\sqrt{4-t^2}}{2} + C \quad 1$$

$$= \sin^{-1}\left(\frac{x^2}{2}\right) - \frac{1}{4}x^2\sqrt{4-x^4} + C. \quad 1$$

6. (**10 points**) Evaluate the following improper integral or show that it diverges

$$\int_1^\infty \frac{1}{(1+x^2)\tan^{-1}x}\,dx$$

$$\int_1^\infty \frac{1}{(1+x^2)\tan^{-1}x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(1+x^2)\tan^{-1}x}\,dx \quad ②$$

$$= \lim_{t\to\infty} \ln|\tan^{-1}x| \Big]_1^t \quad ③$$

$$= \lim_{t\to\infty}\left[\ln|\tan^{-1}t| - \ln|\tan^{-1}1|\right] \quad ①$$

$$= \ln\left(\frac{\pi}{2}\right) - \ln\left(\frac{\pi}{4}\right) \quad 1+1$$

$$= \ln\left(\frac{\pi/2}{\pi/4}\right)$$

$$= \ln 2 \quad ②$$

7. (**10 points**) Evaluate $\displaystyle\int_0^{\ln 2}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 dx.$

$$= \int_0^{\ln 2} \tanh^2 x \, dx \quad ③$$

$$= \int_0^{\ln 2} 1 - \operatorname{sech}^2 x \, dx \quad ②$$

$$= x - \tanh x \Big]_0^{\ln 2} \quad ② = 1+1$$

$$= (\ln 2 - \tanh(\ln 2)) - (0 - 0)$$

$$= \ln 2 - \frac{3}{5} \quad ①$$

$$\bullet\ \tanh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{e^{\ln 2} + e^{-\ln 2}} = \frac{2 - \frac{1}{2}}{2 + \frac{1}{2}} = \frac{3}{5}$$

②

8. (**14 points**) Evaluate $\int e^{2x} \tan^{-1}(e^x)\, dx = I$

Let $t = e^x$. Then $dt = e^x\, dx$. 2

$$I = \int e^x \cdot \tan^{-1}(e^x) \cdot e^x\, dx$$

$$= \int t \cdot \tan^{-1} t \; dt \quad 2$$

$u = \tan^{-1} t$, $dv = t\,dt$

$du = \frac{1}{1+t^2}\,dt$, $v = \frac{1}{2}t^2$

$$= \frac{1}{2}t^2 \cdot \tan^{-1} t - \frac{1}{2}\int \frac{t^2}{t^2+1}\, dt \quad 5$$

$$= \frac{1}{2}t^2 \cdot \tan^{-1} t - \frac{1}{2}\int 1 - \frac{1}{t^2+1}\, dt \quad 2$$

$$= \frac{1}{2}t^2 \cdot \tan^{-1} t - \frac{1}{2}\left[t - \tan^{-1} t\right] + C \quad 2$$

$$= \frac{1}{2}e^{2x} \tan^{-1}(e^x) - \frac{1}{2}e^x + \frac{1}{2}\tan^{-1}(e^x) + C \quad 1$$

OR

$u = \tan^{-1}(e^x)$, $dv = e^{2x}$

$du = \frac{e^x}{1+e^{2x}}\,dx$, $v = \frac{1}{2}e^{2x}$

$$I = \frac{1}{2}e^{2x}\tan^{-1}(e^x) - \frac{1}{2}\int e^{2x}\frac{e^x}{1+e^{2x}}\,dx, \quad 5$$

Let $t = e^x$. Then $dt = e^x\,dx$ 2

$$\begin{cases} -\frac{1}{2}\int \frac{t^2}{1+t^2}\,dt & 2 \\ -\frac{1}{2}\int 1 - \frac{1}{t^2+1}\,dt & 2 \\ -\frac{1}{2}\left[t - \tan^{-1} t\right] & 2 \end{cases}$$

$$= \frac{1}{2}e^{2x}\tan^{-1}(e^x) - \frac{1}{2}e^x + \frac{1}{2}\tan^{-1}(e^x) + C \quad 1$$