King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 101 - Exam I - Term 141

Duration: 90 minutes

Name: Key ID Number: ______

Section Number: ______ Serial Number: ______

Class Time: ______ Instructor's Name: ______

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 7 pages of problems (Total of 6 Problems)

Page Number	Points	Maximum Points
1		7
2		6
3		8
4		7
5		8
6		6
7		8
Total		50

1. (**7 points**) Sketch the graph of a function f that satisfies the following conditions:

(*i*) $f(0) = 0$,

(*ii*) $\lim_{x \to -\infty} f(x) = 1$,

(*iii*) f has a jump discontinuity at $x = -1$,

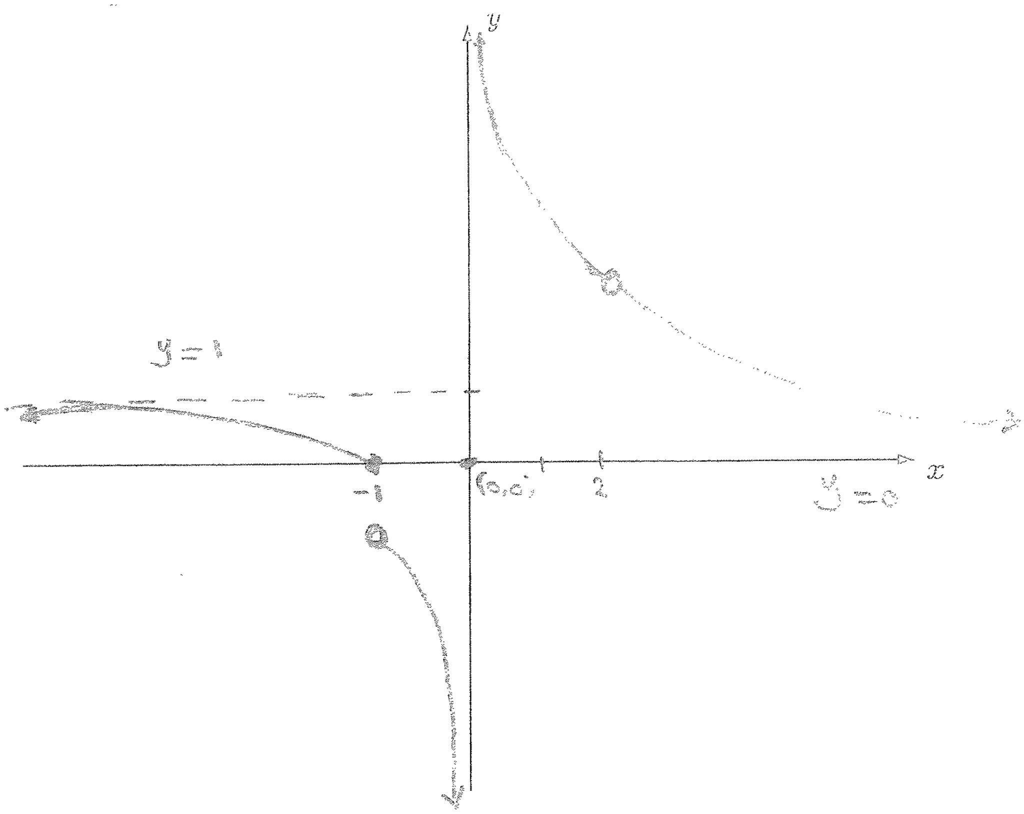
(*iv*) $\lim_{x \to 0^-} f(x) = -\infty$,

(*v*) $\lim_{x \to 0^+} f(x) = \infty$,

(*vi*) f has a removable discontinuity at $x = 2$.

(*vii*) $\lim_{x \to \infty} f(x) = 0$

1-point for each condition.



other graphs are possible.

2. Find the limit if it exists. Justify your work.

a) **(3 points)** $\lim_{x\to 9} \frac{9-x}{3-\sqrt{x}}$

$$\lim_{x\to 9} \frac{9-\sqrt{x}}{3-\sqrt{x}}$$

$$= \lim_{x\to 9} \frac{(3-\sqrt{x})(3+\sqrt{x})}{(3-\sqrt{x})} \quad (2 \text{ pts})$$

$$= \lim_{x\to 9} (3+\sqrt{x}) = 6 \quad (1 \text{ pt})$$

b) **(3 points)** $\lim_{x\to -\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to -\infty} \frac{x}{\sqrt{x^2(1+\frac{1}{x^2})}}$

$$= \lim_{x\to -\infty} \frac{x}{|x|\sqrt{1+\frac{1}{x^2}}} \quad (1 \text{ pt})$$

$$= \lim_{x\to -\infty} \frac{x}{-x\sqrt{1+\frac{1}{x^2}}} \quad (1 \text{ pt})$$

$$= \lim_{x\to -\infty} \frac{1}{-\sqrt{1+\frac{1}{x^2}}}$$

$$= \frac{1}{-1} = -1 \quad (1 \text{ pt})$$

c) (**4 points**) $\lim\limits_{x\to 0} \sin^2 x \cos\left(\frac{1}{x}\right)$

$-1 \le \cos\left(\frac{1}{x}\right) \le 1 \quad , \quad x \ne 0$

$-\sin^2 x \le \sin^2 x \cos\left(\frac{1}{x}\right) \le \sin^2 x \qquad (1\text{ pt})$

Since $\lim\limits_{x\to 0} -\sin^2 x = 0 = \lim\limits_{x\to 0} \sin^2 x \qquad (1\text{ pt})$

, then by the Sequeeze Theorem (1 pt)

$\lim\limits_{x\to 0} \sin^2 x \cos\left(\frac{1}{x}\right) = 0 \qquad (1\text{ pt})$

d) (**4 points**) $\lim\limits_{x\to 0} \dfrac{\sin(3x)\ \cot(5x)}{x\cot(4x)}$

$$= \lim_{x\to 0} \frac{\sin 3x \cdot \dfrac{\cos 5x}{\sin 5x}}{x \cdot \dfrac{\cos 4x}{\sin 4x}}$$

$$= \lim_{x\to 0} 3\underbrace{\frac{\sin 3x}{3x}}_{(1\text{ pt})} \cdot \frac{\frac{1}{5}\,\dfrac{5x}{\sin 5x}}{\frac{1}{4}\,\dfrac{4x}{\sin 4x}} \cdot \frac{\cos 5x}{\cos 4x} \qquad (1\text{ pt}) \quad (1\text{ pt})$$

$$= (3)\left(\frac{4}{5}\right)(1) = \frac{12}{5} \qquad (1\text{ pt})$$

3. (**7 points**)Find the values of a and b so that the following function is continuous everywhere.

$$f(x) = \begin{cases} x^2 - 4x - 2 & \text{if } x < 2 \\ 9x^2 - bx + 4 & \text{if } 2 \le x < 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}$$

i) f is continuous for $x < 2$ (as it is a polynomial)

(1pt) " " " " for $2 < x < 3$ (" ")

" " " " for $x > 3$ (" ")

ii) for f to be continuous at $x = 2$, we must have

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$ (1pt)

$f(2) = 40 - 2b$

$\lim_{x \to 2^-} x^2 - 4x - 2 = -6$, $\lim_{x \to 2^+} (9x^2 - bx + 4) = 40 - 2b$

(1pt)

$\Rightarrow 40 - 2b = -6 \Rightarrow 2b = 46 \Rightarrow b = 23$ (1pt)

iii) at $x = 3$

$\lim_{x \to 3^+} (2x - a + b) = \lim_{x \to 3^-} (9x^2 - bx + 4) = f(3)$ (1pt)

$\Rightarrow 6 - a + b = 85 - 3b$ (1pt)

$\Rightarrow 6 - a + 23 = 85 - 69$

$\Rightarrow a = 13$ (1pt)

4. a) (**4 points**)Use the Intermediate Value Theorem to show that the equation $2 - e^x = \sqrt{x}$ has a root between 0 and 1.

Let $f(x) = 2 - e^x - \sqrt{x}$ (1 pt)

i) f is continuous on $[0,1]$ (1 pt)

ii) $f(0) = 2 - 1 - 0 = 1 > 0$ and (1 pt)

$f(1) = 2 - e - 1 = 1 - e < 0$

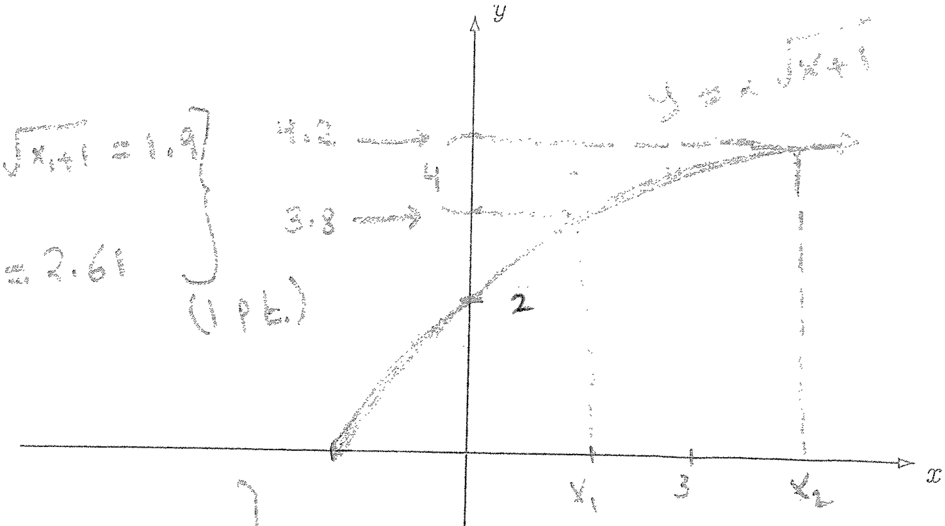
then, by Intermediate Value Theorem there is $0 < c < 1$ such that $f(c) = 0$, i.e (1 pt)

$2 - e^c - \sqrt{c} = 0$ or $2 - e^c = \sqrt{c}$.

b) (**4 points**) Use the graph of $f(x) = 2\sqrt{x+1}$ to find a number $\delta > 0$ such that for all x,

$$0 < |x - 3| < \delta \Rightarrow |f(x) - 4| < 0.2$$

$a = 3$, $L = 4$, $\epsilon = 0.2$



$f(x_1) = 3.8$

$\Rightarrow 2\sqrt{x_1+1} = 3.8 \Rightarrow \sqrt{x_1+1} = 1.9$

$\Rightarrow x_1 + 1 = 3.61 \Rightarrow x_1 = 2.61$ (1 pt)

$f(x_2) = 4.2$

$\Rightarrow 2\sqrt{x_2+1} = 4.2$

$\Rightarrow \sqrt{x_2+1} = 2.1 \Rightarrow x_2 + 1 = 4.41$ 1 pt

$\Rightarrow x_2 = 3.41$

We take $\delta = \min\{3 - 2.61, 3.41 - 3\}$ ← (1 pt)

$= \min\{0.39, 0.41\} = 0.39$ (or any smaller +ve number)

(1 pt)

5. **(6 points)** Find an equation for the tangent to the curve of $g(x) = \dfrac{3}{\sqrt{2x+7}}$ at the point $(1, 1)$. **(You must use limits)**

$$m = \text{Slope} = \lim_{h\to 0} \frac{1}{h}\left[\frac{3}{\sqrt{2(1+h)+7}} - \frac{3}{\sqrt{9}}\right] \quad (1\text{pt})$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{3}{\sqrt{2h+9}} - 1\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{3-\sqrt{2h+9}}{\sqrt{2h+9}}\right] \quad (1\text{pt})$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{9-(2h+9)}{\sqrt{2h+9}\cdot(3+\sqrt{2h+9})}\right] \quad (1\text{pt})$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left[\frac{-2\cancel{h}}{\sqrt{2h+9}\cdot(3+\sqrt{2h+9})}\right]$$

$$= \frac{-2}{(3)(6)} = -\frac{1}{9} \quad (1\text{pt})$$

The equation is given by

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 1 = -\frac{1}{9}(x-1) \quad \text{or}$$

$$y = -\frac{1}{9}x + \frac{10}{9} \quad (2\text{pts})$$

6. a) **(4 points)** Use limits to find all horizontal asymptotes of the graph of $f(x) = \dfrac{|x-1|\,(x+1)}{(x^2-1)}$.

$$\lim_{x\to\infty} \frac{|x-1|(x+1)}{x^2-1} = \lim_{x\to\infty} \frac{(x-1)(x+1)}{x^2-1} = 1 \quad (1\tfrac{1}{2}\text{ pts})$$

$$\lim_{x\to-\infty} \frac{|x-1|(x+1)}{x^2-1} = \lim_{x\to-\infty} \frac{-(x-1)(x+1)}{x^2-1} = -1 \quad (1\tfrac{1}{2}\text{ pts})$$

$\therefore$ $y = 1$ and $y = -1$ are the two horizontal asymptotes. (1 pt)

b) **(4 points)** Use limits to find all vertical asymptotes of the graph of $f(x) = \dfrac{|x-1|}{x(x^2-1)}$.

the zeros of the denominator are $x=0$, $x=\pm 1$

$x=0$:

(1 pt) $$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{-(x-1)}{x(x-1)(x+1)} = -\infty$$

$x=-1$

(1 pt) $$\lim_{x\to -1^+} \frac{-(x-1)}{x(x-1)(x+1)} = +\infty$$

$x=1$: $$\lim_{x\to 1^+} \frac{(x-1)}{x(x-1)(x+1)} = \frac{1}{2}$$

(1 pt) $$\lim_{x\to 1^-} \frac{-(x-1)}{x(x-1)(x+1)} = -\frac{1}{2}$$

(1 pt)

$\therefore$ V.A are $x=0$ and $x=-1$ only