King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 101 - Exam I - Term 142

Duration: 90 minutes

KEY

Name: ____________________ ID Number: ____________________

Section Number: ____________________ Serial Number: ____________________

Class Time: ____________________ Instructor's Name: ____________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 6 pages of problems (Total of 8 Problems)

Question Number	Points	Maximum Points
1		8
2		19
3		10
4		18
5		14
6		10
7		8
8		13
Total		100

1. **(8 points)** Find the value of a that makes the following function

$$f(x) = \begin{cases} a - x & \text{if } x \leq -1 \\ x + 1 & \text{if } x > -1 \end{cases}$$

continuous everywhere.

- If $x < -1$, $f(x) = a - x$ is continuous since it is a polynomial
- If $x > -1$, $f(x) = x + 1$ is continuous since it is a polynomial
- Now f will be continuous at $x = -1$ if

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = f(-1) \quad 3$$

$$\Longrightarrow \lim_{x \to -1^-} (a - x) = \lim_{x \to -1^+} (x + 1) \quad 2$$

$$\Longrightarrow a + 1 = 0 \quad 2$$

$$\Longrightarrow a = -1 \quad 1$$

2. Find each limit or show that it does not exist.

(a) **(5 points)** $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right)$. 2

$$= \lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{x} \right) \quad 1$$, as when $x \to 0^+$, then $x > 0$ and so $|x| = x$

$$= \lim_{x \to 0^+} 0 \quad 1$$

$$= 0 \quad 1$$

b) **(6 points)** $\lim_{t \to 1} \frac{\sin(t^3 - 1)}{3t^2 - 3}$.

$$= \lim_{t \to 1} \frac{\sin(t^3 - 1)}{t^3 - 1} \cdot \frac{t^3 - 1}{3(t^2 - 1)} \quad 2$$

$$= \lim_{t \to 1} \frac{\sin(t^3 - 1)}{t^3 - 1} \cdot \frac{(t - 1)(t^2 + t + 1)}{3(t - 1)(t + 1)} \quad 1$$

$$= \lim_{t \to 1} \frac{\sin(t^3 - 1)}{t^3 - 1} \cdot \frac{t^2 + t + 1}{3(t + 1)} \quad 1$$

1

$$= 1 \cdot \frac{3}{3 \cdot 2}$$

$$= \frac{1}{2} \quad 1$$

c) **(8 points)** $\lim_{x\to-\infty} (2x + \sqrt{4x^2+3x-2})$.

1 $$= \lim_{x\to-\infty} \left(2x+\sqrt{4x^2+3x-2}\right)\cdot \frac{2x-\sqrt{4x^2+3x-2}}{2x-\sqrt{4x^2+3x-2}}$$

2 $$= \lim_{x\to-\infty} \frac{-3x+2}{2x-\sqrt{4x^2+3x-2}}$$

1 $$= \lim_{x\to-\infty} \frac{-3x+2}{2x-|x|\sqrt{4+\frac{3}{x}-\frac{2}{x^2}}}$$

1 $$= \lim_{x\to-\infty} \frac{-3x+2}{2x+x\sqrt{4+\frac{3}{x}-\frac{2}{x^2}}}, \quad x\to-\infty \Rightarrow |x|=-x$$

1 $$= \lim_{x\to-\infty} \frac{x\left(-3+\frac{2}{x}\right)}{x\left(2+\sqrt{4+\frac{3}{x}-\frac{2}{x^2}}\right)}$$

$$= \lim_{x\to-\infty} \frac{-3+\frac{2}{x}}{2+\sqrt{4+\frac{3}{x}-\frac{2}{x^2}}}$$ 1

$$= \frac{-3+0}{2+\sqrt{4+0-0}}$$

$$= \frac{-3}{4}$$ 1

3. **(10 points)** Find a number k so that the following limit

$$\lim_{x\to-2} \frac{3x^2+kx+k+3}{x^2+x-2}$$

exists. Then find the limit for the found value of k.

- Note that Deno. $\to 0$ as $x\to-2$. If Num $\not\to 0$ when $x\to-2$, then then the given limit equals $\pm\infty$ (i.e, DNE). So for the limit to exists we must have

$$\lim_{x\to-2} 3x^2+kx+k+3=0$$ 3

i.e, $$12-2k+k+3=0$$

$$\Longrightarrow \quad k=15$$ 2

- when $k=15$, we have

$$\lim_{x\to-2} \frac{3x^2+15x+18}{x^2+x-2} = \lim_{x\to-2} \frac{3(x+2)(x+3)}{(x+2)(x-1)}$$ 2

$$= \lim_{x\to-2} \frac{3(x+3)}{(x-1)}$$ 1

$$= \frac{3(1)}{-3}$$

$$= -1$$ 2

4. Let $f(x) = \dfrac{\ln(2x+1)}{x^2-4}$

a) (**8 points**) Where is f continuous?

It is continuous in its domain:

- $2x+1 > 0 \Rightarrow x > -\frac{1}{2}$ 2
- $x^2-4 \neq 0 \Rightarrow x \neq \pm 2$ 2

So f is continuous on $(-\frac{1}{2}, 2) \cup (2, \infty)$. 4

b) (**10 points**) Find all vertical asymptotes of f.
Justify your answer using limits.

- We check $x = -\frac{1}{2}$ (as it is a V.A. for $\ln(2x+1)$)

$$\lim_{x \to -\frac{1}{2}^+} \frac{\ln(2x+1)}{x^2-4} = \frac{-\infty}{\frac{1}{4}-4} = \infty$$ 4

- We check the zeros of the denominator: $x = \pm 2$. But since $x = -2$ is not on the boundary of the domain (see part (a)), then we check only $x = 2$.

$$\lim_{x \to 2^+} \frac{\ln(2x+1)}{x^2-4} = \infty$$ 4

So the V.A. are

$x = -\frac{1}{2}$ and $x = 2$. 1 + 1

5. Consider the curve $y = \dfrac{1}{x+1}$, $x < 0$, $x \neq -1$.

a)(**8 points**) Use limits to find the slope of the curve at any point $x = a \neq -1$.

$$\text{Slope} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \quad 2$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{a+h+1} - \frac{1}{a+1}\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(a+h+1)(a+1)} \quad 2$$

$$= \lim_{h\to 0} \frac{-1}{(a+h+1)(a+1)} \quad 1$$

$$= \frac{-1}{(a+1)(a+1)} \quad 1$$

$$= \frac{-1}{(a+1)^2} \quad 2 \quad , \quad a \neq -1$$

b) (**6 points**) Find all points at which the slope equals $-\dfrac{1}{4}$.

$$\text{Slope} = -\frac{1}{4}$$

$$\Rightarrow \frac{-1}{(a+1)^2} = -\frac{1}{4} \quad 1 \quad , \text{ from part (a)}$$

$$\Rightarrow (a+1)^2 = 4$$

$$\Rightarrow a+1 = \pm 2$$

$$\Rightarrow a = -3 \text{ or } a = 1 \quad 1$$

$$\Rightarrow a = -3 \quad \text{since } x < 0 \quad 1$$

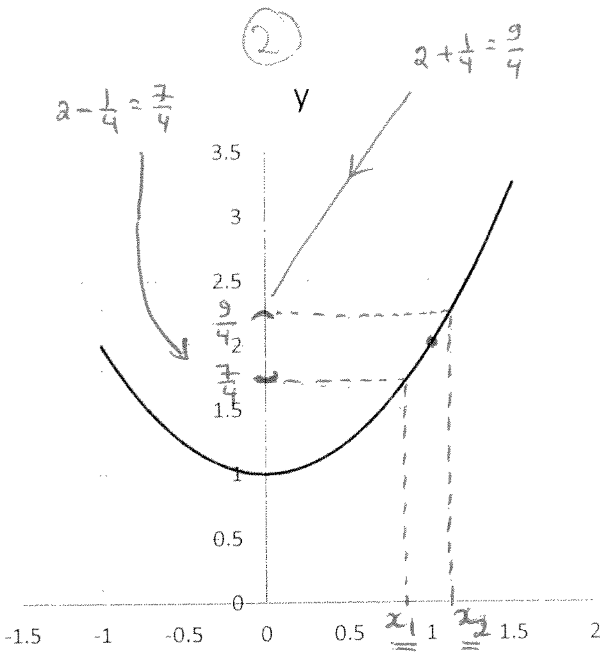
The required point is $\left(-3, -\frac{1}{2}\right)$. 1

6. (**10 points**) Use the **graph** of $f(x) = x^2 + 1$ to find a number $\delta > 0$ such that for all x,

$$0 < |x - 1| < \delta \Rightarrow |f(x) - 2| < \frac{1}{4}.$$

[Note that $\sqrt{2} \approx 1.41$, $\sqrt{3} \approx 1.73$, $\sqrt{5} \approx 2.24$, $\sqrt{7} \approx 2.65$]

- $\lim_{x \to 1} f(x) = 2$, $\varepsilon = \frac{1}{4}$



From the graph

- $f(x_1) = \frac{7}{4} \Longrightarrow x_1^2 + 1 = \frac{7}{4} \Longrightarrow x_1^2 = \frac{3}{4}$ 1

$\Longrightarrow x_1 = \frac{\sqrt{3}}{2}$ 1

- $f(x_2) = \frac{9}{4} \Longrightarrow x_2^2 + 1 = \frac{9}{4} \Longrightarrow x_2^2 = \frac{5}{4}$ 1

$\Longrightarrow x_2 = \frac{\sqrt{5}}{2}$ 1

- $\delta_1 = 1 - x_1 = 1 - \frac{\sqrt{3}}{2} \approx 1 - 0.865 = 0.135$ 1
- $\delta_2 = x_2 - 1 = \frac{\sqrt{5}}{2} - 1 \approx 1.12 - 1 = 0.12$ 1

, we may choose

$\delta = \min(\delta_1, \delta_2)$ 1

$= \min(0.135, 0.12) = 0.12$ 1

(or any smaller positive number)

7. (**8 points**) Use the **Intermediate Value Theorem** to show that the equation $x^x = 3x - 1$ has a positive solution.

2 Let $f(x) = x^x - 3x + 1$ $\qquad (x^x = e^{x \ln x})$

1 Choose the interval $[1, 3]$

Since

- f is continuous on $[1, 3]$ 1
- $f(1) = 1^1 - 3(1) + 1 = -1 < 0$ 1
- $f(3) = 3^3 - 3(3) + 1 = 19 > 0$ 1

then, by IVT, there is a number $c \in (1, 3)$ such that $\Rightarrow \boxed{c > 0}$

$$f(c) = 0$$

2 i.e, $c^c = 3c - 1$

So the given equation has a positive solution.

8. (**13 points**) Sketch the graph of a function f that satisfies the following conditions:

(i) $\lim_{x \to -\infty} f(x) = 0$ 2

(ii) $\lim_{x \to 0^-} f(x) = 1$ 2

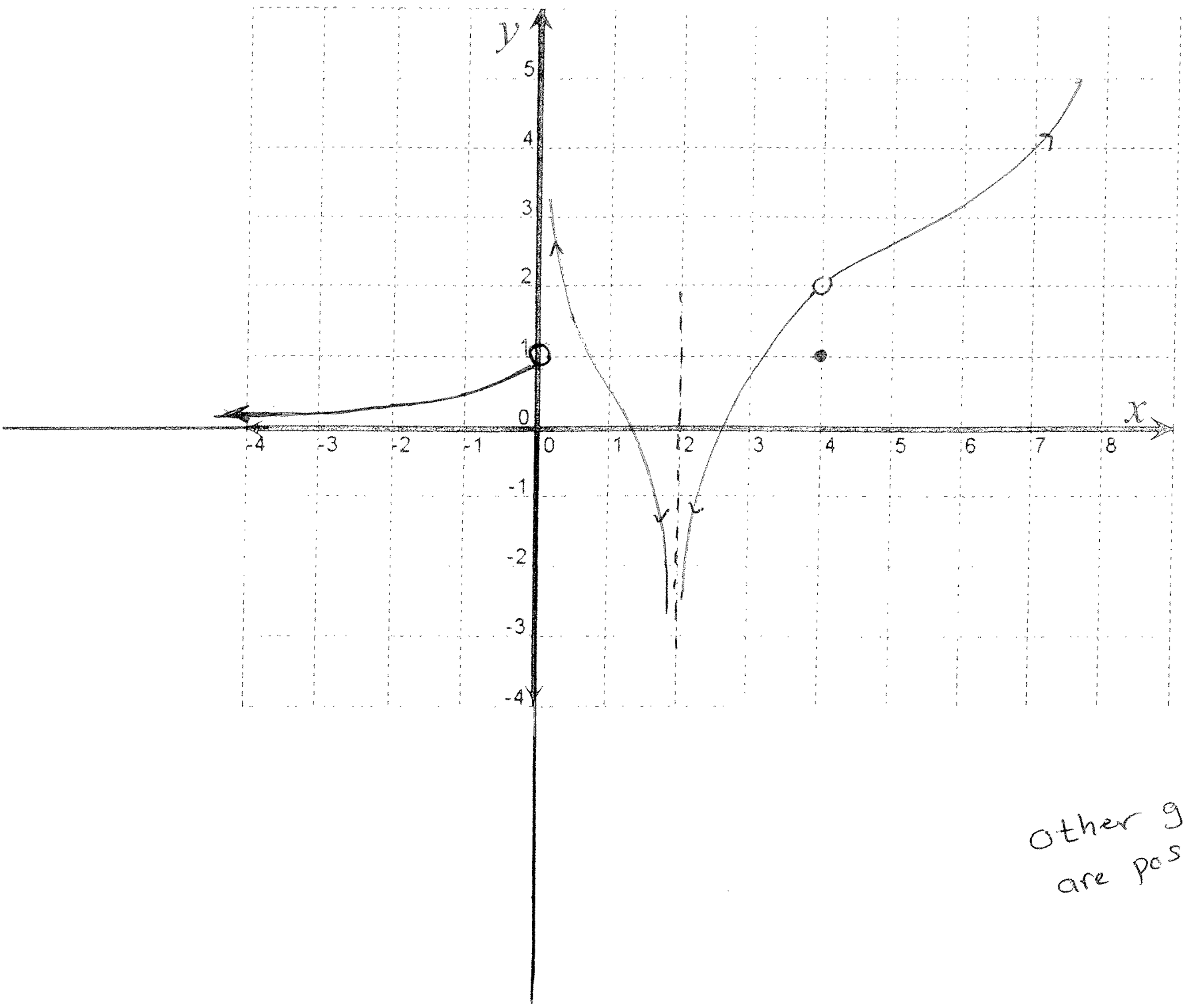
(iii) $\lim_{x \to 0^+} f(x) = \infty$ 2

(iv) $\lim_{x \to 2} f(x) = -\infty$ 2

(v) f has a removable discontinuity at $x = 4$. 2

(vi) $f(4) = 1$ 1

(vii) $\lim_{x \to \infty} f(x) = \infty$ 2



Other graphs are possible.