## Formula for AS483 Actuarial Risk Theory & Credibility

### Chap 3 Basic Distributional Quantitites

**3.1** *Moments*:

| Quantity | | Continuous | Discrete |
|---|---|---|---|
| $k^{th}$ *Raw* moment | $\mu'_k = E[X^k]$ | $\int_{-\infty}^{\infty} x^k f(x)dx$ | $\sum_j x_j^k p(x_j)$ |
| $k^{th}$ *Central* moment | $\mu_k = E[(X-\mu)^k]$ | $\int_{-\infty}^{\infty} (x-\mu)^k f(x)dx$ | $\sum_j (x_j-\mu)^k p(x_j)$ |
| $k^{th}$ *Excess* Loss moment | $e_X^k(d) = E[(X-d)^k \mid X>d]$ | $\dfrac{\int_d^{\infty} (x-d)^k f(x)dx}{1-F(d)}$ | $\dfrac{\sum_{x_j>d} (x_j-\mu)^k p(x_j)}{1-F(d)}$ |
| $k^{th}$ *Left* Censored & shifted moment | $E[(X-d)_+^k] = e_X^k(d)\,[1-F(d)]$ | $\int_d^{\infty} (x-d)^k f(x)dx$ | $\sum_{x_j>d} (x_j-d)^k p(x_j)$ |
| $k^{th}$ *Limited* loss moment | $E[(X\wedge u)^k]$ | $\int_{-\infty}^{u} x^k f(x)dx + u^k S(u)$ | $\sum_{x_j\le u} x_j^k p(x_j) + u^k S(u)$ |

**3.2** $100pth$ **percentile** $\pi_p$ of a random variable is such that $F(\pi_p-) \le p \le F(\pi_p)$. $\pi_{0.5}$ is the **median**.

**3.3** *Generating Functions and Sums of Random Variables*:

Moment generating function (mgf): $M_X(t) = E[e^{tX}]$ Probability generating function (mgf): $P_X(z) = E[z^X]$.

$M_X(z) = P_X(e^z)$ $P_X(z) = M_X(\ln z)$

| Quantity | *Expectation* | *Variance* | $M_{S_k}(t)$ | $P_{S_k}(z)$ | $\lim_{k\to\infty} \dfrac{S_k - E[S_k]}{\sqrt{Var(S_k)}}$ |
|---|---|---|---|---|---|
| $S_k = \sum_{i=1}^{k} X_i$ (independent $X_i s$) | $E[S_k] = \sum_{i=1}^{k} E[X_i]$ | $Var(S_k) = \sum_{i=1}^{k} Var(X_i)$ | $\prod_{i=1}^{k} M_{X_i}(t)$ | $\prod_{i=1}^{k} P_{X_i}(z)$ | *approx* $N(0,1)$ |
| $S_k = \sum_{i=1}^{k} X_i$ (dependent $X_i s$) | $E[S_k] = \sum_{i=1}^{k} E[X_i]$ | $Var(S_k) = \sum_{i=1}^{k} Var(X_i) + 2\sum_{i=1}^{k}\sum_{j=1}^{i} Cov(X_i, X_j)$ | $M_{S_k}(t)$ | $P_{S_k}(z)$ | |

**3.4** *Tails of Distributions*:

Classifications based on: (1) moments (2) limiting tail behavior (3) hazard rate functions 4) Mean Excess loss functions

1) $E[X^k]$ 2) $\lim_{x\to\infty} \dfrac{S_1(x)}{S_2(x)} = \lim_{x\to\infty} \dfrac{S'_1(x)}{S'_2(x)} = \lim_{x\to\infty} \dfrac{-f_1(x)}{-f_2(x)} = \lim_{x\to\infty} \dfrac{f_1(x)}{f_2(x)}$

3) $h(x)$ 4) $e(d) = \dfrac{\int_d^{\infty} S(x)dx}{S(d)} = \displaystyle\int_0^{\infty} \dfrac{S(y+d)}{S(d)} dy$

3.4.5) Equilibrium distributions and tail behavior $f_e(x) = \dfrac{S(x)}{E(X)}$, $x \ge 0$.

$$h_e(x) = \frac{f_e(x)}{S_e(x)} = \frac{S(x)}{\int_x^{\infty} S(t)dt} = \frac{1}{e(x)}$$

**3.5** *Risk Measures*:

**Coherent risk measure** $\rho(X)$ has 4 properties below:

1) Subadditivity: $\rho(X+Y) \le \rho(X) + \rho(Y)$ 2) Monotonicity: if $X \le Y$, then $\rho(X) \le \rho(Y)$

3) Positive homogeneity: if $c > 0$, then $\rho(cX) \le c\rho(X)$

4) Translation invariance: if $c > 0$, then $\rho(X+c) = \rho(X) + c$.

**Standard deviation principle**: $Pr(X > \mu + k\sigma)$ =exceedance probability.

**Value-at-Risk**, $VaR_p(X) = \pi_p$. violates subadditivity.

**Tail-Value-at-Risk,** $TVaR_p(X) = E[X|X > \pi_p] = \dfrac{\int_{\pi_p}^{\infty} xf(x)dx}{1-F(\pi_p)} = \pi_p + e_X(\pi_p) = \pi_p + \dfrac{E[X] - E[X\wedge \pi_p]}{1-p}$

### Chap 4 Characteristics of Actuarial Models

**4.2.2** *Parametric Distribution Families*:

Def4.1 A **parametric** distribution is a set of distribution functions determined by specifying one or more values called parameters. The number of parameters is fixed and finite.

Def4.2 A **scale distribution** if, when a random variable from that set of parametric distributions is multiplied by a positive constant, the resulting random variable is also in that set of distributions.

Def4.3 For random variables with nonnegative support, a **scale parameter** for a scale distribution meets two conditions: (1) when a member of the scale distribution is multiplied by a positive constant, the scale parameter is multiplied by the same constant. (2) when a member of the scale distribution is multiplied by a positive constant, all other parameters are unchanged.

Def4.4 A parametric **distribution family** is a set of parametric distributions that are related in some meaningful way.

**4.2.3** *Finite Mixture Distributions*:

Def4.5 A random variable $Y$ is a **$k$-point mixture** of the random variables $X_1, X_2, \ldots, X_k$ if its cdf is given by $F_Y(y) = \sum_{i=1}^{k} a_i F_{X_i}(y)$ where all $a_i > 0$ and $\sum_{i=1}^{k} a_i = 1$.

Def4.6 A **variable-component mixture** distribution has a distribution function that can be written as $F_Y(y) = \sum_{j=1}^{K} a_j F_{X_j}(y)$ where $\sum_{j=1}^{K} a_j = 1$, all $a_j > 0$ $j = 1, 2, \cdots, K$, $K = 1, 2, \cdots$.

**4.2.4** *Data-dependent Distributions*:

Def4.7 A **data-dependent distribution** is at least as complex as the data or knowledge that produced it, and the number of "parameters" increases as the number of data points or amount of knowledge increases.

Def4.8 The **empirical model** is a discrete distribution based on a sample of size $n$ that assigns probability $1/n$ to each data point.

**Chap 5 Continuous Actuarial Models**

**5.2** *Creating New Distributions*:

| Transformation | | $F_Y(y)$ | $f_Y(y)$ |
|---|---|---|---|
| *Multiplication* by a constant | $Y = \theta X$ | $F_X(y/\theta)$ | $\frac{1}{\theta} f_X(y/\theta)$ |
| Raising to a *power*<br>*a) Inverse*<br>*b) Inverse Transformed* | $Y = X^{1/\tau},\ \tau > 0$<br>$Y = X^{-1}$<br>$Y = X^{1/\tau}$, $\tau < 0, \tau \neq -1$ | $F_X(y^\tau)$<br>$1 - F_X(y^{-1})$<br>$1 - F_X(y^\tau)$ | $\tau y^{\tau-1} f_X(y^\tau)$<br>$-f_X(y^{-1})$<br>$-\tau y^{-\tau-1} f_X(y^{-\tau})$ |
| *Exponentiation* | $Y = e^X$ | $F_X(\ln y)$ | $\frac{1}{y} f_X(\ln y)$ |
| *Mixing* | $Y\|\Lambda$ with $f_\Lambda(\lambda)$ | $\int F_{Y\|\Lambda}(y\|\lambda) f_\Lambda(\lambda) d\lambda$<br>$E[X^k] = E\left[E[X^k\|\Lambda]\right]$ | $\int f_{X\|\Lambda}(x\|\lambda) f_\Lambda(\lambda) d\lambda$<br>$Var(X) = E\left[Var[X\|\Lambda]\right] + Var(E[X\|\Lambda])$ |
| *Frailty Model* | $Y\|\Lambda$ with $h_{Y\|\Lambda}(y\|\Lambda) = \Lambda a(y)$ | $1 - S_{Y\|\Lambda}(y\|\lambda) = 1 - e^{-\lambda A(y)}$<br>$1 - S_Y(y) = 1 - E[e^{-\Lambda A(y)}]$ | $\frac{d}{dy} E[e^{-\Lambda A(y)}]$ |
| *Splicing* | $f_i(y) = pdf$<br>$\sum_{i=1}^{k} a_i = 1$ | | $\begin{cases} a_1 f_1(y) & c_0 < y < c_1 \\ a_2 f_2(y) & c_1 < y < c_2 \\ \vdots & \vdots \\ a_k f_k(y) & c_{k-1} < y < c_k \end{cases}$ |

**5.3.2** *Selected Distributions and Their Relationships*:

a) 2-parametric family: (1) Transformed Beta family (2) Inverse/Transformed gamma family.
b) limiting distribution c) Heavy-tailed distribution

**5.4** *Linear Exponential Family*: $f(x;\theta) = \dfrac{p(x)e^{r(\theta)x}}{q(\theta)}$. $E[X] = \mu(\theta) = \dfrac{q'(\theta)}{r'(\theta)q(\theta)}$. $Var(X) = \upsilon(\theta) = \dfrac{\mu'(\theta)}{r'(\theta)}$.

**Chap 6 Discrete Distributions**

| $(a,b,0)$ *Distribution* | $p_k = \Pr(N = k)$ | $P(z) = P_N(z) = E[z^N] = \sum_{k=0}^{\infty} p_k z^k$ | $E[N]$ | $Var(N)$ | *par* $a$ | $b$ | $N = \sum_{i=1}^{n} N_i$ |
|---|---|---|---|---|---|---|---|
| *Poisson* | $\frac{e^{-\lambda}\lambda^k}{k!}$, $\lambda > 0,\ k = 0, 1, \cdots$ | $e^{\lambda(z-1)}$ | $\lambda$ | $\lambda$ | $0$ | $\lambda$ | $Poi(\sum_{i=1}^{n} \lambda_i)$ |
| *Negative Binomial* | $\binom{k+r-1}{k}\left(\frac{1}{1+\beta}\right)^r \left(\frac{\beta}{1+\beta}\right)^k$<br>$r > 0,\ \beta > 0,\ k = 0, 1, \cdots$ | $[1 - \beta(z-1)]^{-r}$ | $r\beta$ | $r\beta(1+\beta)$ | $\frac{\beta}{1+\beta}$ | $\frac{(r-1)\beta}{1+\beta}$ | |
| *Binomial* | $\binom{m}{k} q^k (1-q)^{m-k}$,<br>$0 < q < 1,\ k = 0, 1, \cdots, m$ | $[1 + q(z-1)]^m$ | $mq$ | $mq(1-q)$ | $\frac{-q}{1-q}$ | $\frac{(m+1)q}{1-q}$ | $B\left(\sum_{i=1}^{n} m_i, q\right)$ |

Geometric is Negative Binomial with $r = 1$.

**6.6** *Truncation and Modification at zero: the (a,b,1) class*

| class | recursive | $k$ | subclass | $p_0$ | $p_k$ | $pgf$ |
|---|---|---|---|---|---|---|
| $a,b,0$ | $\frac{p_k}{p_{k-1}} = a + \frac{b}{k}$ | $k = 1, 2, \cdots$ | | $p_0$ | $p_k$ | $P(z)$ |
| $a,b,1$ | $\frac{p_k}{p_{k-1}} = a + \frac{b}{k}$ | $k = 2, 3, \cdots$ | $(ZT)$ *Zero–Truncated* | $p_0^T = 0$ | $p_k^T = \frac{p_k}{1-p_0}$ | $P^T(z) = \frac{P(z)-p_0}{1-p_0}$ |
| | | | $(ZM)$ *Zero–Modified* | $p_0^M > 0$ | $p_k^M = (1-p_0^M)p_k^T$ | $P^M(z) = p_0^M + (1-p_0^M)P^T(z)$ |

| Distribution | $p_0$ | $a$ | $b$ | Parameter space |
|---|---|---|---|---|
| Poisson | $e^{-\lambda}$ | 0 | $\lambda$ | $\lambda > 0$ |
| ZT Poisson | 0 | 0 | $\lambda$ | $\lambda > 0$ |
| ZM Poisson | *arbitrary* | 0 | $\lambda$ | $\lambda > 0$ |
| Binomial | $(1-q)^m$ | $-\frac{q}{(1-q)}$ | $\frac{(m+1)q}{(1-q)}$ | $0 < q < 1$ |
| ZT Binomial | 0 | $-\frac{q}{(1-q)}$ | $\frac{(m+1)q}{(1-q)}$ | $0 < q < 1$ |
| ZM Binomial | *arbitrary* | $-\frac{q}{(1-q)}$ | $\frac{(m+1)q}{(1-q)}$ | $0 < q < 1$ |
| Negative Binomial (NB) | $(1+\beta)^{-r}$ | $\frac{\beta}{1+\beta}$ | $\frac{(r-1)\beta}{1+\beta}$ | $r > 0, \beta > 0$ |
| ETNB | 0 | $\frac{\beta}{1+\beta}$ | $\frac{(r-1)\beta}{1+\beta}$ | $r > -1, r \neq 0, \beta > 0$ |
| ZM ETNB | *arbitrary* | $\frac{\beta}{1+\beta}$ | $\frac{(r-1)\beta}{1+\beta}$ | $r > -1, r \neq 0, \beta > 0$ |
| Geometric | $(1+\beta)^{-1}$ | $\frac{\beta}{1+\beta}$ | 0 | $\beta > 0$ |
| ZT geometric | 0 | $\frac{\beta}{1+\beta}$ | 0 | $\beta > 0$ |
| ZM geometric | *arbitrary* | $\frac{\beta}{1+\beta}$ | 0 | $\beta > 0$ |
| Logarithmic | 0 | $\frac{\beta}{1+\beta}$ | $\frac{-\beta}{1+\beta}$ | $\beta > 0$ |
| ZM Logarithmic | *arbitrary* | $\frac{\beta}{1+\beta}$ | $\frac{-\beta}{1+\beta}$ | $\beta > 0$ |

**8.2** *Definitions of deductibles*

| Type | variable | definition | pdf | sdf |
|---|---|---|---|---|
| *ordinary* deductible | per-payment $Y^P = Y^L \vert Y^L > 0$ | $Y^P = \begin{cases} undefined & X \leq d \\ X-d & X > d. \end{cases}$ | $f_{Y^P}(y) = \frac{f_X(y+d)}{S_X(d)}$ | $S_{Y^P}(y) = \frac{S_X(y+d)}{S_X(d)}$ |
| | per-loss | $Y^L = \begin{cases} 0 & X \leq d \\ X-d & X > d. \end{cases}$ | $f_{Y^L}(y) = f_X(y+d)$ | $S_{Y^L}(y) = S_X(y+d)$ |
| *Franchise* deductible | per-payment $Y^P = Y^L \vert Y^L > 0$ | $Y^P = \begin{cases} undefined & X \leq d \\ X & X > d. \end{cases}$ | $f_{Y^P}(y) = \frac{f_X(y)}{S_X(d)} \quad y > d$ | $S_{Y^P}(y) = \begin{cases} 1 & y \leq d \\ \frac{S_X(y)}{S_X(d)} & y > d \end{cases}$ |
| | per-loss | $Y^L = \begin{cases} 0 & X \leq d \\ X & X > d. \end{cases}$ | $f_{Y^L}(y) = \begin{cases} F_X(d) & y = 0 \\ f_X(y) & y > d \end{cases}$ | $S_{Y^L}(y) = \begin{cases} S_X(d) & y \leq d \\ S_X(y) & y > d \end{cases}$ |

| Type | variable | cdf | hazard | Exp loss |
|---|---|---|---|---|
| *ordinary* deductible | per-payment $Y^P = Y^L \| Y^L > 0$ | $F_{Y^P}(y) = \dfrac{F_X(y+d) - F_X(d)}{S_X(d)}$ | $h_{Y^P}(y) = \dfrac{f_X(y+d)}{S_X(y+d)}$ | $E[X] - E[X \wedge d]$ |
| | per-loss | $F_{Y^L}(y) = F_X(y+d)$ | $h_{Y^L}(y)$ | $\dfrac{E[X] - E[X \wedge d]}{1 - F(d)}$ |
| *Franchise* deductible | per-payment $Y^P = Y^L \| Y^L > 0$ | $F_{Y^P}(y) = \begin{cases} 0 & y \le d \\ \dfrac{F_X(y) - F_X(d)}{S_X(d)} & y > d \end{cases}$ | $h_{Y^P}(y) = \begin{cases} 0 & y \le d \\ h_X(y) & y > d \end{cases}$ | $E[X] - E[X \wedge d]$ $+d(1 - F(d))$ |
| | per-loss | $F_{Y^L}(y) = \begin{cases} F_X(d) & y \le d \\ F_X(y) & y > d \end{cases}$ | $h_{Y^L}(y) = \begin{cases} 0 & y \le d \\ h_X(y) & y > d \end{cases}$ | $\dfrac{E[X] - E[X \wedge d]}{1 - F(d)} + d$ |

**8.3** *Loss Elimination Ratio*

| | *ordinary* deductible |
|---|---|
| Loss Elimination ratio | $\dfrac{E[X] - (E[X] - E[X \wedge d])}{E[X]} = \dfrac{E[X \wedge d]}{E[X]}$ |
| Inflation effects | $r$ =inflation rate |
| E[cost per-loss] | $(1+r)\,(E[X] - E[X \wedge d/(1+r)])$ |
| E[cost per-payment] | $\dfrac{(1+r)\,(E[X] - E[X \wedge d/(1+r)])}{1 - F(d/(1+r))}$ |

**8.4** *Policy Limits*

| Type | definition | pdf | cdf | E[cost after inflation] |
|---|---|---|---|---|
| *Policy* limits | $Y = \begin{cases} X & X < u \\ u & X \ge u. \end{cases}$ | $f_Y(y) = \begin{cases} f_X(y) & y < u \\ 1 - F_X(u) & y = u \end{cases}$ | $F_Y(y) = \begin{cases} F_X(y) & y < u \\ 1 & y \ge u. \end{cases}$ | $(1+r)E[X \wedge u/(1+r)]$ |

**8.5** *Coinsurance, deductibles, and policy Limits*

$$Y^L = \begin{cases} 0 & X < \dfrac{d}{1+r} \\ \alpha[(1+r)X - d] & \dfrac{d}{1+r} \le X < \dfrac{u}{1+r} \\ \alpha\,(u - d) & X \ge \dfrac{u}{1+r}. \end{cases}$$

| Type | Per-loss | Per-payment |
|---|---|---|
| 1st moment | $E[Y^L] = \alpha(1+r)\left(E[X \wedge \dfrac{u}{1+r}] - E[X \wedge \dfrac{d}{1+r}]\right)$ | $E[Y^P] = \dfrac{E[Y^L]}{1 - F_X\left(\dfrac{d}{1+r}\right)}$ |
| 2nd moment | $E[(Y^L)^2] = \alpha^2(1+r)^2(E\left[\left(X \wedge \dfrac{u}{1+r}\right)^2\right] - E\left[\left(X \wedge \dfrac{d}{1+r}\right)^2\right] - 2\dfrac{d}{1+r}E[X \wedge \dfrac{u}{1+r}] + 2\dfrac{d}{1+r}E[X \wedge \dfrac{d}{1+r}])$ | |

**8.6** *Impact of deductibles on claim frequency* $\upsilon = \Pr(X > d) = 1 - F_X(d)$

| Distribution for $N^L$ | Parameter for $N^P$ | |
|---|---|---|
| Poisson | $\lambda^* = \upsilon\lambda$ | |
| ZM Poisson | $\lambda^* = \upsilon\lambda$ | $p_0^{M*} = \dfrac{p_0^M - e^{-\lambda} + e^{-\upsilon\lambda} - p_0^M e^{-\upsilon\lambda}}{1 - e^{-\lambda}}$ |
| Binomial | $q^* = \upsilon q$ | |
| ZM Binomial | $q^* = \upsilon q$ | $p_0^{M*} = \dfrac{p_0^M - (1-q)^m + (1-\upsilon q)^m - p_0^M(1-\upsilon q)^m}{1 - (1-q)^m}$ |
| Negative Binomial (NB) | $\beta^* = \upsilon\beta$ | $r^* = r$ |
| ZM Negative Binomial | $\beta^* = \upsilon\beta$, $r^* = r$ | $p_0^{M*} = \dfrac{p_0^M - (1+\beta)^{-r} + (1+\upsilon\beta)^{-r} - p_0^M(1+\upsilon\beta)^{-r}}{1 - (1+\beta)^{-r}}$ |
| ZM Logarithmic | $\beta^* = \upsilon\beta$ | $p_0^{M*} = 1 - (1 - p_0^M)\dfrac{\ln(1+\upsilon\beta)}{\ln(1+\beta)}$ |

**9.1 Aggregate** *Loss Models*

| Model | Collective Risk | Individual risk |
|---|---|---|
| definition | $S=\sum_{j=1}^{N} X_j,\ \ j=1,2,\cdots,N$ | $S=\sum_{j=1}^{n} X_j,\ \ j=1,2,\cdots,n$ |
| Assumptions | 1) Conditioned on $N=n$, $X_j$ are i.i.d. r.v.<br>2) Conditioned on $N=n$, the common distribution of $X_j$ does not depend on $n$.<br>3) The distribution of $N$, does not depend on values of $X_j$ | $n$ is fixed |

**9.2** *Model Choices*

Prefer (1) Scale distributions for severity distribution (2) Models with pgf $P_N(z;\alpha)=Q(z)^\alpha$ for frequency distribution (3) infinitely divisible (i.e., $\alpha^*=(1+r)\alpha$, $r$ =increase rate of business volume, $r>-1$) frequency distributions (4) zero-modified even if not in form of (2) above.

**9.3** *Compound model for aggregate claims S*

3 modeling steps (1) Develop model for the *distribution* of $N$ from data (2) Develop model for *common distribution* of $X_j$ from data (3) from (1) and (2), Calculate the distribution of $S$.

Compound distribution $S$: $F_S(x)=\Pr(S\le x)=\sum_{n=0}^{\infty} p_n F_X^{*n}(x)$, $F_X^{*n}(x)$ =*n-fold* convolution of cdf of$X$.

| Model | Discrete $X$ | Continuous $X$ |
|---|---|---|
| $F_X^{*k}(x)$ | $\sum_{y=0}^{x} F_X^{*(k-1)}(x-y)f_X(y)$<br>$x=0,1,\cdots,\ k=2,3,\cdots$<br>$F_X^{*1}(x)=F_X(x)$ | $\int_0^x F_X^{*(k-1)}(x-y)f_X(y)dy$<br>$k=2,3,\cdots$<br>$F_X^{*1}(x)=F_X(x)$ |
| $f_X^{*k}(x)$ | $\sum_{y=0}^{x} f_X^{*(k-1)}(x-y)f_X(y)$<br>$x=0,1,\cdots,\ k=2,3,\cdots$<br>$f_X^{*1}(x)=f_X(x)$ | $\int_0^x f_X^{*(k-1)}(x-y)f_X(y)dy$<br>$k=2,3,\cdots$<br>$f_X^{*1}(x)=f_X(x)$ |
| $f_S(x)$ | $P(S=x)=\sum_{n=0}^{\infty} p_n f_X^{*n}(x)$<br>$x=0,1,\cdots,$ | $\begin{cases} P(S=0)=p_0 & x=0 \\ \sum_{n=1}^{\infty} p_n f_X^{*n}(x) & x>0 \end{cases}$ |
| $P_S(z)$ | $E[z^S]=E[P_X(z)^N]=P_N\left[P_X(z)\right]$ | |
| $M_S(z)$ | $P_N\left[M_X(z)\right]$ | $P_N\left[M_X(z)\right]$ |
| $E[S]$ | $\mu'_{S1}=\mu'_{N1}\mu'_{X1}=E[N]E[X]$ | $\mu'_{S1}=\mu'_{N1}\mu'_{X1}=E[N]E[X]$ |
| $Var(S)$ | $\mu_{S2}=\mu'_{N1}\mu_{X2}+\mu_{N2}\left(\mu'_{X1}\right)^2$<br>$=E[N]Var[X]+Var[N]\left(E[X]\right)^2$ | $\mu_{S2}=\mu'_{N1}\mu_{X2}+\mu_{N2}\left(\mu'_{X1}\right)^2$<br>$=E[N]Var[X]+Var[N]\left(E[X]\right)^2$ |
| $E[(S-E[S])^3]$ | $\mu_{S3}=\mu'_{N1}\mu_{X3}+3\mu_{N2}\mu'_{X1}\mu_{X2}+\mu_{N3}\left(\mu'_{X1}\right)^3$ | $\mu_{S3}=\mu'_{N1}\mu_{X3}+3\mu_{N2}\mu'_{X1}\mu_{X2}+\mu_{N3}\left(\mu'_{X1}\right)^3$ |

| net stop-loss premium | If $\Pr(a<S<b)=0,\ a\le d\le b$ | If $\Pr(S=kh)=f_k,\ \ h>0,k=0,1,\cdots$<br>$\Pr(S=x)=0\ \ $ *elsewhere* |
|---|---|---|
| $E[(S-d)_+]$ | $\frac{b-d}{b-a}E[(S-a)_+]+\frac{d-a}{b-a}E[(S-b)_+]$<br>linear interpolation | $h\sum_{m=0}^{\infty}(1-F_S[(m+j)h])$<br>$d=jh,\ \ \ j=0,1,\cdots$<br>$E[(S-(j+1)h)_+]=$<br>$E[(S-jh)_+]-h(1-F_S[jh])$ |

**9.4** *Analytic results for some compound distributions*

| $S=\sum_{j=0}^{N} X_j$ | with *Exponential* severity | Compound Negative binomial-exponential | Compound Poisson |
|---|---|---|---|
| Frequency $N$ | general | Negative binomial $(r,\beta)$ | $Poisson(\lambda)$ $\lambda=\sum_{j=1}^{n}\lambda_j$ |
| Severity $X$ | exponential$(\theta)$ | exponential$(\theta)$ | $F_X(x)=\sum_{j=1}^{n}\frac{\lambda_j}{\lambda}F_j(x)$ |
| $M_S(z)$ | $(1-\theta z)^{-n}$ | $P_N[M_X(z)]=P_N[(1-\theta z)^{-1}]$ $=\left(1-\beta\left[(1-\theta z)^{-1}-1\right]\right)^{-r}$ $=\left(1+\frac{\beta}{1+\beta}\left[(1-\theta\,(1+\beta)\,z)^{-1}-1\right]\right)^{-r}$ | $e^{\left(\lambda\left[\sum_{j=1}^{n}\frac{\lambda_j}{\lambda}M_j(z)-1\right]\right)}$ |
| $F_X^{*n}(x)$ | $\Gamma(n;x/\theta)$ $=1-\sum_{j=0}^{n-1}\frac{y^j e^{-y}}{j!}$ $=1-\sum_{n=1}^{\infty}p_n\sum_{j=0}^{n-1}\frac{y^j e^{-y}}{j!}$ $=1-e^{-y}\sum_{j=0}^{n-1}\frac{y^j}{j!}\sum_{n=j+1}^{\infty}p_n$ $=1-e^{-y}\sum_{j=0}^{n-1}\overline{P}_j\frac{y^j}{j!}$ $y=x/\theta,\ n$ integer $\overline{P}_j=\sum_{n=j+1}^{\infty}p_n,\ j=0,1,\cdots$ | | |
| $F_S(x)$ | $p_0+\sum_{n=1}^{\infty}p_n\Gamma\left(n;\frac{x}{\theta}\right)$ | $1-\sum_{n=1}^{r}\binom{r}{n}\left(\frac{\beta}{1+\beta}\right)^n\left(\frac{1}{1+\beta}\right)^{r-n}$ $\times\sum_{j=0}^{n-1}\frac{(x\theta^{-1}(1+\beta)^{-1})^j e^{-x/(\theta(1+\beta))}}{j!}$ if $r=1,$ compound geometric -exponential distribution $1-\frac{\beta}{1+\beta}e^{-x/(\theta(1+\beta))}$ | |
| $f_S(x)$ | $\sum_{n=1}^{\infty}p_n\frac{x^{n-1}e^{-x/\theta}}{\theta^n\Gamma(n)}$ | $\begin{cases}1/(1+\beta) & x=0 \text{ if } r=1,\\ \frac{\beta}{\theta\,(1+\beta)^2}e^{-x/(\theta(1+\beta))} & x>0 \text{ if } r=1.\end{cases}$ | |

**9.5** *Computing aggregate claims distribution*

(1) Approximating distribution (2) direct calculation by *n-fold* Convolution (3) recursive method

**9.6** *Recursive methods*

| $S=\sum_{j=0}^{N} X_j$ | with $(a,b,1)$ frequency | with $(a,b,0)$ frequency |
|---|---|---|
| Frequency $N$ | $p_k=\left(a+\frac{b}{k}\right)p_{k-1},$ $k=2,3,4,\cdots$ | $p_k=\left(a+\frac{b}{k}\right)p_{k-1},$ $k=1,2,3,4,\cdots$ |
| Severity $X$ | $f_X(x)$ $x=0,1,\cdots$ monetary units | $f_X(x)$ $x=0,1,\cdots$ monetary units |
| $f_S(x),$ $x\wedge m=\min(x,m)$ | $\frac{[p_1-(a+b)p_0]\,f_X(x)+\sum_{y=1}^{x\wedge m}(a+by/x)f_X(y)f_S(x-y)}{1-af_X(0)}$ | $\frac{\sum_{y=1}^{x\wedge m}(a+by/x)f_X(y)f_S(x-y)}{1-af_X(0)}$ |

| Distribution for $N$ | $f_S(0)$ |
|---|---|
| Poisson | $\exp(\lambda[f_0 - 1])$ |
| Geometric | $[1+\beta(1-f_0)]^{-1}$ |
| Binomial | $[1+q(f_0-1)]^{m}$ |
| Negative Binomial (NB) | $[1+\beta(1-f_0)]^{-r}$ |
| ZM Poisson | $p_0^M + (1-p_0^M)\dfrac{e^{\lambda f_0}-1}{e^{\lambda}-1}$ |
| ZM Geometric | $p_0^M + (1-p_0^M)\dfrac{f_0}{1+\beta(1-f_0)}$ |
| ZM Binomial | $p_0^M + (1-p_0^M)\dfrac{[1+q(f_0-1)]^m-(1-q)^m}{1-(1-q)^m}$ |
| ZM Negative Binomial | $p_0^M + (1-p_0^M)\dfrac{[1+\beta(1-f_0)]^{-r}-(1+\beta)^{-r}}{1-(1+\beta)^{-r}}$ |
| ZM Logarithmic | $p_0^M + (1-p_0^M)\left(1-\dfrac{\ln[1+\beta(1-f_0)]}{\ln(1+\beta)}\right)$ |

**9.6.2** Underflow/overflow problems

**9.6.3** Numerical Stability

**9.6.4** *Continuous severity with (a, b,1) frequency class*

Volterra Integral equation of the 2nd kind: $f_S(x) = p_1 f_X(x) + \int_0^x \left(a + b\frac{y}{x}\right) f_X(y) f_S(x-y)dy$

**9.6.5** *Constructing Arithmetic Distributions*

Discretizing (arithmetizing) a continuous distribution: 2 methods.

| | Method of rounding (mass dispersal) | Method of local moment matching |
|---|---|---|
| rationale | round to midpoint of interval $h$ = span | matches $p$ moments of severity distribution. |
| $f_0$ | $\Pr\left(X < \frac{h}{2}\right) = F_X\left(\frac{h}{2} - 0\right)$ | |
| $f_j$ $j = 1, 2, \cdots$ | $\Pr\left(jh - \frac{h}{2} \le X < jh + \frac{h}{2}\right)$ $= F_X\left(jh + \frac{h}{2}\right) - F_X\left(jh - \frac{h}{2}\right)$ | |
| last point $f_m$ | $1 - F_X[(m-0.5)h - 0]$ | |
| System of $p+1$ equations. $r = 0, 1, 2, \cdots, p$ | | $\sum_{j=0}^{p}(x_k + jh)^r m_j^k = \int_{x_k-0}^{x_k+ph-0} x^r dF_X(x)$ $-0$ = discrete probability at $x_k$ included discrete probability at $x_k + ph$ excluded |
| $m_j^k$ $j = 0, 1, \cdots, p$ | | $\int_{x_k-0}^{x_k+ph-0} \prod_{i\neq j} \frac{x - x_k - ih}{(j-i)h} dF_X(x)$ |

**9.7** Impact of individual policy modifications on aggregate payments

$F_{Y^L}(y) = (1-\upsilon) + \upsilon F_{Y^P}(y),\ y \ge 0,\ \ 1-\upsilon = \Pr(Y^L = 0) = F_{Y^L}(0)$

$M_{Y^L}(z) = (1-\upsilon) + \upsilon M_{Y^P}(z) \rightarrow E[e^{zY^L}] = E[e^{zY^L}|Y^L = 0]\Pr(Y^L = 0) + E[e^{zY^L}|Y^L > 0]\Pr(Y^L > 0)$

$P_{N^P}(z) = P_{N^L}(1 - \upsilon + \upsilon z)$

| | per-loss | per-payment |
|---|---|---|
| $S =$ | $\begin{cases} 0 & N^L = 0 \\ Y_1^L + Y_2^L + \cdots + Y_{N^L}^L & \end{cases}$ | $\begin{cases} 0 & N^P = 0 \\ Y_1^P + Y_2^P + \cdots + Y_{N^P}^P & \end{cases}$ |
| $M_S(z)$ | $E[e^{zS}] = P_{N^L}(M_{Y^L}(z))$ | $E[e^{zS}] = P_{N^P}(M_{Y^P}(z))$ |

$P_{N^L}(M_{Y^L}(z)) = P_{N^L}((1-\upsilon) + \upsilon M_{Y^P}(z)) = P_{N^P}(M_{Y^P}(z))$

**9.8** *Individual Risk Model* $S = \sum_{j=1}^{n} X_j$

| | life-insurance | other insurance |
|---|---|---|
| $f_{X_j}(x)$ | $\begin{cases} 1-q_j & x=0 \\ q_j & x=b_j \end{cases}$ | $X_j = I_j B_j$ |
| $f_{I_j}(i)$ | | $\begin{cases} 1-q_j & i=0 \\ q_j & i=1 \end{cases}$ |
| $f_{B_j}(i)$ | | any distribution |
| $E[S]$ | $\sum_{j=1}^{n} b_j q_j$ | $\sum_{j=1}^{n} E[B_j]q_j = \sum_{j=1}^{n} \mu_j q_j$ |
| $Var(S)$ | $\sum_{j=1}^{n} b_j^2 q_j (1-q_j)$ | $\sum_{j=1}^{n} \left[Var[B_j]q_j + E[B_j]^2 q_j (1-q_j)\right]$ $= \sum_{j=1}^{n} \left[\sigma_j^2 q_j + \mu_j^2 q_j (1-q_j)\right]$ |
| $P_S(z)$ | $\begin{cases} \prod_{j=1}^{n} \left(1-q_j+q_j z^{b_j}\right) \\ (1+q(z-1))^n & if\ b_j=1,\ q_j=q \end{cases}$ | |
| $M_S(z)$ | | $\prod_{j=1}^{n} \left(1-q_j+q_j M_{B_j}(z)\right)$ |

**9.8.2** *Parametric approximation*

Normal Approximation $\Pr(S > s_0) \doteq \Pr\left(Z > \dfrac{s_0 - E[S]}{\sqrt{Var(S)}}\right)$

## Chap **10 Review of Mathematical Statistics**

**10.2** *Point Estimation*

| Quality Measures | Definition |
|---|---|
| unbiasedness | $E[\widehat{\theta}\vert\theta] = \theta$, for all $\theta$ |
| $\text{bias}_{\widehat{\theta}}(\theta)$ | $E[\widehat{\theta}\vert\theta] - \theta$ |
| Asymptotic unbiasedness | $\lim_{n\to\infty} E[\widehat{\theta}_n\vert\theta] = \theta$, for all $\theta$ |
| consistency (*weak* consistency) | $\lim_{n\to\infty} \Pr\left(\left\vert\widehat{\theta}_n - \theta\right\vert > \delta\right) = 0$, for all $\delta > 0$ and any $\theta$ |
| Mean Square Error $MSE_{\widehat{\theta}}(\theta)$ | $E[\left(\widehat{\theta} - \theta\right)^2 \vert\theta]$ $= Var(\widehat{\theta}\vert\theta) + \left(bias_{\widehat{\theta}}(\theta)\right)^2$ |
| Uniformly minimum variance unbiased estimator (UMVUE) | unbiased and smallest variance |

**10.3** *Interval Estimation*

A $100(1-\alpha)\%$ confidence interval for parameter $\theta$ is a pair of random values, $L$ and $U$, computed from a random sample such that $\Pr(L \le \theta \le U) \ge 1-\alpha$ for all $\theta$.

**10.4** *Hypothesis tests*

Hypothesis: Null $H_0$ Alternative $H_1$

Significance level $\alpha$: Pr(Type 1 error)=max Pr(reject $H_0|H_0$ is true).

Uniformly most powerful hypothesis test: test with smallest $\alpha$ and smaller Pr(Type 2 error).

$p$-value: Pr(Test statistic in disagreement with $H_0|H_0$ is true)

| Rule | Decision |
|---|---|
| Test statistic in CR | reject $H_0$ |
| $p$-value$< \alpha$ | reject $H_0$ |

## **Chap 14** *Frequentist Estimation for Discrete Distribution Models*

$$L = \prod_{k=0}^{\infty} p_k^{n_k} \qquad l = \sum_{k=0}^{\infty} n_k \ln p_k$$

| Model | Method of Moments | $l$ | $mle$ | $Variance$ |
|---|---|---|---|---|
| 14.1 Poisson | $\lambda=\overline{x}=\frac{\sum_{k=0}^{\infty}kn_k}{\sum_{k=0}^{\infty}n_k}$ | $-\lambda n+\sum_{k=0}^{\infty}kn_k\ln\lambda-\sum_{k=0}^{\infty}n_k\ln k!$ | $\widehat{\lambda}=\frac{1}{n}\sum_{k=0}^{\infty}kn_k=\overline{x}$ | $Var(\widehat{\lambda})=\frac{\lambda}{n}$ |
| 14.2 Negative Binomial | $r\beta=\overline{x}$<br>$r\beta(1+\beta)=s^2$ | $\sum_{k=0}^{\infty}n_k\ln\binom{r+k-1}{k}$<br>$-\sum_{k=0}^{\infty}n_k\left[(r+k)\ln(1+\beta)-k\ln\beta\right]$ | $\widehat{\beta}=\overline{x}/\widehat{r}$<br>$\widehat{r}$ iteratively solved | |
| 14.3 Binomial | $m\widehat{q}=\frac{\sum_{k=0}^{m}kn_k}{\sum_{k=0}^{m}n_k}$ | $\sum_{k=0}^{m}n_k\left[\ln\binom{m}{k}+k\ln q\right]$<br>$+\sum_{k=0}^{m}n_k(m-k)\ln(1-q)$ | $\widehat{q}=\frac{1}{m}\frac{\sum_{k=0}^{m}kn_k}{\sum_{k=0}^{m}n_k}$ | |
| 14.4 $(a,b,1)$ class | | $l_0+l_1$,<br>$l_0=n_0\ln p_0^M+\sum_{k=1}^{\infty}n_k\ln\left(1-p_0^M\right)$<br>$l_1=\sum_{k=1}^{\infty}n_k\left[\ln p_k-\ln\left(1-p_0\right)\right]$ | $\widehat{p}_0^M=\frac{n_0}{n}$<br>$a,b$ from $\max(l_1)$ | |
| 14.5 Compound Models | | $l=\sum_{k=0}^{\infty}n_k\ln g_k,\ g_0=P_1(0)$<br>$g_k=$ probability of a compound distribution | | |

14.6 Exposure Effects on Maximum Likelihood Estimation
$\lambda=$poisson parameter for single exposure. $e_k=$year $k$ exposures.
$m=\#$ of years in data.
$n_k=$number of claims. Then, number of claims has Poisson distribution with parameter $\lambda e_k$.

$$L=\prod_{k=1}^{m}\frac{e^{-\lambda e_k}(\lambda e_k)^{n_k}}{n_k!},\qquad l=\sum_{k=1}^{m}\left[-\lambda e_k+n_k\ln(\lambda e_k)-\ln(n_k!)\right],\qquad \widehat{\lambda}=\frac{\sum_{k=1}^{m}n_k}{\sum_{k=1}^{m}e_k}$$

**Chap 17 Introduction and Limited Fluctuation Credibility**

Credibility: how much trust (or belief) you can put on a policyholder's past claims. $Z=$credibility factor

| **Credibility Situation** | **Past claims $\bar{X}$** | **Manual rate $M$** | **Credibility Premium $P_c$** |
|---|---|---|---|
| *Zero* Credibility | 0% | 100% | $P_c=M$ |
| *Partial* | $Z*100\%$ | $(1-Z)*100\%$ | $P_c=Z\bar{X}+(1-Z)M$ |
| *Full* Credibility | 100% | 0% | $P_c=\bar{X}$ |

**17.3** *Full Credibility*:

$$\Pr(-r\xi\le\bar{X}-\xi\le r\xi)\ge p.\quad(17.1)\qquad \Pr\left(\left|\frac{\bar{X}-\xi}{\sigma/\sqrt{n}}\right|\le\frac{r\xi\sqrt{n}}{\sigma}\right)\ge p\qquad y_p=\inf_y\left\{\Pr\left(\left|\frac{\bar{X}-\xi}{\sigma/\sqrt{n}}\right|\le y\right)\ge p\right\}\quad(17.2)$$

$$\Pr\left(\left|\frac{\bar{X}-\xi}{\sigma/\sqrt{n}}\right|\le y_p\right)=p\quad(17.3)\qquad \frac{\sigma}{\xi}\le\frac{r}{y_p}\sqrt{n}=\sqrt{\frac{n}{\lambda_0}},\quad(17.4)$$

$$Var(\bar{X})=\frac{\sigma^2}{n}\le\frac{\xi^2}{\lambda_0}\quad(17.5)\qquad n\ge\lambda_0\left(\frac{\sigma}{\xi}\right)^2.\quad(17.6)\qquad y_p\approx\Phi^{-1}\left(\frac{1+p}{2}\right).$$ usual $r=0.05$, $p=0.9$.

**17.4** *Partial Credibility*:

$$P_c=Z\bar{X}+(1-Z)M,\quad(17.7)\qquad Z=\frac{n}{n+k},\quad(17.8)\qquad Z=\min\left\{\frac{\xi}{\sigma}\sqrt{\frac{n}{\lambda_0}},1\right\}=\min\left(\sqrt{\frac{n}{\lambda_0\sigma^2/\xi^2}},1\right)\quad(17.9)$$

**Chap 18 Introduction and Limited Fluctuation Credibility**

**18.2** *Conditional Distributions and Expectation*

$f_X(\mathrm{x})=\int f_{X|Y}(x|y)f_Y(y)dy$ (18.1) $E(X|Y=y)=\int xf_{X|Y}(x|y)dx$ (18.2)

$E[E(X|Y)]=E(X)$ (18.3) $E\left\{E[h(X,Y)|Y]\right\}=E[h(X,Y)]$ (18.4)

$Var(X|Y)=E\left\{[X-E(X|Y)]^2|Y\right\}$ (18.5) $Var(X)=E[Var(X|Y)]+Var[E(X|Y)]$ (18.6)

**18.3** *The Bayesian Methodology*

$f_{\mathbf{X}}(\mathbf{x}) = \int \left[ \prod_{j=1}^{n} f_{X_j|\Theta}(x_j|\theta) \right] \pi(\theta) d\theta \qquad (18.7) \qquad f_{X_{n+1}|\mathbf{X}}(x_{n+1}|\mathbf{x}) = \frac{1}{f_{\mathbf{X}}(\mathbf{x})} \int \left[ \prod_{j=1}^{n+1} f_{X_j|\Theta}(x_j|\theta) \right] \pi(\theta) d\theta \qquad (18.8)$

$\pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x}) = \frac{f_{\mathbf{X},\Theta}(\mathbf{x},\theta)}{f_{\mathbf{X}}(\mathbf{x})} = \frac{1}{f_{\mathbf{X}}(\mathbf{x})} \int \left[ \prod_{j=1}^{n} f_{X_j|\Theta}(x_j|\theta) \right] \pi(\theta) \qquad (18.9)$

$f_{X_{n+1}|\mathbf{X}}(x_{n+1}|\mathbf{x}) = f_{X_{n+1}|\Theta}(x_{n+1}|\theta) \pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x}) d\theta \qquad (18.10)$

Hypothetical Mean: $\mu_{n+1}(\theta) = E(X_{n+1}|\Theta = \theta) = \int x_{n+1} f_{X_{n+1}|\Theta}(x_{n+1}|\theta) dx_{n+1} \qquad (18.11)$

Bayesian Premium: $E(X_{n+1}|\mathbf{X} = \mathbf{x}) = \int x_{n+1} f_{X_{n+1}|\mathbf{X}}(x_{n+1}|\mathbf{x}) dx_{n+1} = \int \mu_{n+1}(\theta) \pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x}) d\theta \qquad (18.12\&13)$

**18.4** *Credibility Premium*

$Q = E\left\{ \left[ \mu_{n+1}(\Theta) - \alpha_0 - \sum_{j=1}^{n} \alpha_j X_j \right]^2 \right\} \qquad (18.14) \qquad E(X_{n+1}) = \widetilde{\alpha}_0 + \sum_{j=1}^{n} \widetilde{\alpha}_j E(X_j) \qquad (18.15)$

Credibility Premium: $P_c = \widetilde{\alpha}_0 + \sum_{j=1}^{n} \widetilde{\alpha}_j X_j \qquad (18.18)$

$Q_1 = E\left\{ \left[ E(X_{n+1}|\mathbf{X}) - \alpha_0 - \sum_{j=1}^{n} \alpha_j X_j \right]^2 \right\} \qquad (18.19) \qquad Q_2 = E\left[ \left( X_{n+1} - \alpha_0 - \sum_{j=1}^{n} \alpha_j X_j \right)^2 \right] \qquad (18.20$

**18.5** *The Bühlmann model*

$\mu = E[\mu(\Theta)] \qquad (18.21) \qquad \upsilon = E[\upsilon(\Theta)] \qquad (18.22) \qquad a = Var[\mu(\Theta)] \qquad (18.23)$

$E(X_j) = E[E(X_j|\Theta)] = E[\mu(\Theta)] = \mu \qquad (18.24)$

$Var(X_j) = E[Var(X_j|\Theta)] + Var[E(X_j|\Theta)] = E[\upsilon(\Theta)] + Var[\mu(\Theta)] = \upsilon + a. \qquad (18.25)$

$$\begin{aligned} Cov(X_i, X_j) &= E(X_i, X_j) - E(X_i)E(X_j) = E[E(X_i, X_j|\Theta)] - \mu^2 = E[E(X_i|\Theta)E(X_j|\Theta)] - \{E[\mu(\Theta)]\}^2 \\ &= E\{[\mu(\Theta)]^2\} - \{E[\mu(\Theta)]\}^2 = Var[\mu(\Theta)] = a \qquad (18.26) \end{aligned}$$

Credibility Premium: $P_c = \widetilde{\alpha}_0 + \sum_{j=1}^{n} \widetilde{\alpha}_j X_j = ZX + (1-Z)\mu \qquad (18.27)$

$Z = \frac{n}{n+k} \qquad (18.28) \qquad k = \frac{\upsilon}{a} = \frac{E[Var(X_j|\Theta)]}{Var[E(X_j|\Theta)]} \qquad (18.29)$

**18.6** *The Bühlmann-Straub model* $\quad m = \sum_{i=1}^{n} m_i$

$\sum_{j=1}^{n_i} \widetilde{\alpha}_j = 1 - \frac{\widetilde{\alpha}_0}{\mu} \qquad (18.30) \qquad \widetilde{\alpha}_i = \frac{a}{\upsilon} m_i \left( 1 - \sum_{j=1}^{n} \widetilde{\alpha}_j \right) = \frac{a}{\upsilon} \frac{\widetilde{\alpha}_0}{\mu} m_i \qquad i = 1, ...n \qquad (18.31)$

$\widetilde{\alpha}_0 + \sum_{j=1}^{n} \widetilde{\alpha}_j X_j = Z\bar{X} + (1-Z)\mu \qquad (18.32) \qquad \bar{X} = \sum_{j=1}^{n} \frac{m_j}{m} X_j \qquad (18.33)$

$Var(\bar{X}|\theta) = \sum_{j=1}^{n} \frac{m_j^2}{m^2} = \frac{\upsilon(\theta)}{m}, \qquad Z = \frac{1}{1 + \upsilon/(am)} = \frac{m}{m + \upsilon/(a)} \qquad (18.34)$

**18.7** *Exact Credibility - if Bayes premium=credibility premium*

Linear exponential family: $\mu(\theta) = E(X_j|\Theta = \theta) = \frac{q'(\theta)}{r'(\theta) q(\theta)} \qquad (18.35)$

$\pi(\theta) = \frac{[q(\theta)]^{-k} e^{\mu k r(\theta)} r'(\theta)}{c(\mu, k)}, \qquad \theta_0 < \theta < \theta_1 \qquad (18.36)$

$\frac{d}{d\theta} = \left[ \frac{\pi(\theta)}{r'(\theta)} \right] = -k[\mu(\theta) - \mu]\pi(\theta). \qquad (18.37) \qquad E[\mu(\Theta)] = \mu + \frac{\pi(\theta_0)}{kr'(\theta_0)} - \frac{\pi(\theta_1)}{kr'(\theta_1)} \qquad (18.38)$

If $\frac{\pi(\theta_1)}{r'(\theta_1)} = \frac{\pi(\theta_0)}{r'(\theta_0)} \qquad (18.39) \rightarrow \qquad E[\mu(\Theta)] = \mu \qquad (18.40)$

Posterior distribution in Buhlmann situation: $f_{X_j|\Theta}(x_j|\theta) = \frac{p(x_j) e^{r(\theta) x_j}}{q(\theta)}$

$\pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x}) = \frac{[q(\theta)]^{-k_*} e^{\mu_* k_* r(\theta)} r'(\theta)}{c(\mu_*, k_*)} \qquad (18.41) \qquad k_* = k + n \qquad (18.42)$

$\mu_* = \frac{\mu k + n\bar{x}}{k+n} \qquad (18.43) \qquad E(X_{n+1}|X = x) = \mu_* + \frac{\pi_{\Theta|\mathbf{X}}(\theta_0|\mathbf{x})}{k_* r'(\theta_0)} - \frac{\pi_{\Theta|\mathbf{X}}(\theta_1|\mathbf{x})}{k_* r'(\theta_1)} \qquad (18.44)$

If $\frac{\pi_{\Theta|\mathbf{X}}(\theta_0|\mathbf{x})}{r'(\theta_0)} = \frac{\pi_{\Theta|\mathbf{X}}(\theta_1|\mathbf{x})}{r'(\theta_1)} \qquad (18.45) \rightarrow \qquad E(X_{n+1}|\mathbf{X} = \mathbf{x}) = \mu_* = \frac{\mu k + n\bar{x}}{k+n} \qquad (18.46)$

For $f_{X_j|\Theta}(x_j|\theta) = \theta e^{-\theta x_j} \qquad \pi(\theta) = \frac{\theta^k e^{-\mu k \theta}}{\int_{\theta_0}^{\theta_1} t^k e^{-\mu k t} dt}, \qquad \theta_0 < \theta < \theta_1 \qquad (18.47)$

$$E[\mu(\Theta)] = \mu + \frac{\theta_1^k e^{-\mu k \theta_1} - \theta_0^k e^{-\mu k \theta_0}}{k\int_{\theta_0}^{\theta_1} t^k e^{-\mu k t} dt} \quad (18.48) \qquad E(X_{n+1}|\mathbf{X}=\mathbf{x}) = \mu_* + \frac{\theta_1^{k_*} e^{-\mu_* k_* \theta_1} - \theta_0^{k_*} e^{-\mu_* k_* \theta_0}}{k_* \int_{\theta_0}^{\theta_1} t^{k_*} e^{-\mu_* k_* t} dt} \quad (18.49)$$

$$\pi(\theta) = \frac{\mu k (\mu k \theta)^k e^{-\mu k \theta}}{\Gamma(k+1)}, \quad \theta > 0 \quad (18.50)$$

$$k = \frac{E[\upsilon(\Theta) + [\mu(\theta_0) - \mu]\frac{\pi(\theta_0)}{r'(\theta_0)} - [\mu(\theta_1) - \mu]\frac{\pi(\theta_1)}{r'(\theta_1)}}{E\{[\mu(\Theta) - \mu]^2\}} = \frac{E[\upsilon(\Theta) + \mu(\theta_0)\frac{\pi(\theta_0)}{r'(\theta_0)} - \mu(\theta_1)\frac{\pi(\theta_1)}{r'(\theta_1)}}{Var\left[\mu(\Theta)\right]} \quad (18.51\&52)$$

**Chap 19 Empirical Bayes Parameter Estimation**

**19.1** *Introduction*

$$\bar{X} = \frac{1}{m}\sum_{i=1}^{r} m_i \bar{X}_i = \frac{1}{m}\sum_{i=1}^{r}\sum_{j=1}^{n_i} m_{ij} X_{ij} \quad (19.1) \qquad P_{c_i} = Z_i \bar{X}_i + (1 - Z_i)\mu, \quad i = 1, ...r, \quad (19.2)$$

$$\widehat{P}_{c_i} = \hat{Z}_i \bar{X}_i + (1 - \hat{Z}_i)\hat{\mu}, \quad (19.3) \qquad \hat{Z}_i = \frac{m_i}{m_i + \hat{\upsilon}/\hat{a}}$$

**19.2** *Nonparametric estimation*

$$\sum_{j=1}^{k}\left(Y_j - \bar{Y}\right)^2 = \sum_{j=1}^{k}\left(Y_j - \mu\right)^2 - k\left(\bar{Y} - \mu\right)^2. \quad (19.4) \qquad E\left[\frac{1}{k-1}\sum_{j=1}^{k}\left(Y_j - \bar{Y}\right)^2\right] = \sigma^2 \quad (19.5)$$

$$\hat{\upsilon}_i = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_{ij} - \bar{X}_i\right)^2 \quad (19.6) \qquad \hat{\upsilon} = \frac{1}{r}\sum_{i=1}^{r}\hat{\upsilon}_i = \frac{1}{r(n-1)}\sum_{i=1}^{r}\sum_{j=1}^{n}\left(X_{ij} - \bar{X}_i\right)^2 \quad (19.7)$$

$$\hat{a} = \frac{1}{r-1}\sum_{i=1}^{r}\left(\bar{X}_i - \bar{X}\right)^2 - \frac{\hat{\upsilon}}{n} = \frac{1}{r-1}\sum_{i=1}^{r}\left(\bar{X}_i - \bar{X}\right)^2 - \frac{1}{rn(n-1)}\sum_{i=1}^{r}\sum_{j=1}^{n}\left(X_{ij} - \bar{X}_i\right)^2 \quad (19.8)$$

$$\hat{\mu} = \bar{X} \quad (19.9) \qquad \sum_{j=1}^{k} m_j\left(X_j - \overline{X}\right)^2 = \sum_{j=1}^{k} m_j\left(X_j - \mu\right)^2 - m\left(\overline{X} - \mu\right)^2. \quad (19.10)$$

$$E\left[\sum_{j=1}^{n} m_j\left(X_j - \bar{X}\right)^2\right] = \beta\left(m - m^{-1}\sum_{j=1}^{n} m_j^2\right) + \alpha(n-1) \quad (19.11) \qquad \hat{\upsilon}_i = \frac{\sum_{j=1}^{n_i} m_{ij}\left(X_{ij} - \bar{X}_i\right)^2}{n_i - 1}, \quad i = 1, ...r, \quad (19.12)$$

$$\hat{\upsilon} = \frac{\sum_{i=1}^{r}\sum_{j=1}^{n_i} m_{ij}\left(X_{ij} - \bar{X}_i\right)^2}{\sum_{i=1}^{r} n_i - 1} \quad (19.13)$$

$$Var\left(\bar{X}_i\right) = Var[E(\bar{X}_i|\Theta_i)] + E[Var(\bar{X}_i|\Theta_i)] = Var[\mu\left(\Theta_i\right)] + E\left[\frac{\upsilon\left(\Theta_i\right)}{m_i}\right] = a + \frac{\upsilon}{m_i} \quad (19.14)$$

$$\hat{a} = \left(m - m^{-1}\sum_{i=1}^{r} m_i^2\right)^{-1}\left[\sum_{i=1}^{r} m_i\left(\bar{X}_i - \bar{X}\right)^2 - \hat{\upsilon}(r-1)\right] \quad (19.15)$$

$$TL = \textstyle\sum_{i=1}^{r} m_i \bar{X}_i \qquad TP = \sum_{i=1}^{r} m_i\left(\hat{Z}_i \bar{X}_i + (1 - \hat{Z}_i)\hat{\mu}\right) = \sum_{i=1}^{r} m_i \bar{X}_i + \sum_{i=1}^{r} m_i \frac{\widehat{k}}{m_i + \widehat{k}}\left(\hat{\mu} - \bar{X}_i\right)$$

$$\text{If } TL = TP \rightarrow \hat{\mu} = \frac{\sum_{i=1}^{r}\hat{Z}_i \bar{X}_i}{\sum_{i=1}^{r}\hat{Z}_i} \quad (19.16)$$

**Chap 20 Simulation**

**20.1** *Simulation Approach*

Step 1) Build a model of $S$ that depends on random variables r.v. $X$, $Y$, $Z$,...., where their distributions and dependencies are known (chap 5 & 9

Step 2) For $j = 1, ..., n$ generate pseudo-random r.v. $x_j$, $y_j$, $z_j$,...., and then compute $s_j$ using model in step (1)

Step 3) The cdf of $S$ may be approximated by $F_n(s)$, the empirical cdf based on the pseudo-random samples $s_1, ..., s_n$

Step 4) Compute quantities of interest, such as mean, variance, percentiles, or probabilities, using empirical cdf.

**20.2.1** *Discrete mixtures* $F_Y(y) = a_1 F_{X_1}(y) + a_2 F_{X_2}(y) + ..... + a_k F_{X_k}(y)$.

It may be difficult to invert this function, but it may be easy to invert the individual $cdfs$. This suggest a two-step process for simulating from a mixture distribution.

1. Simulate a value from the discrete random variable $J$ where $\Pr(J = j) = a_j$.
2. Use an appropriate method (usually inversion) to simulate an observation from a random variable with distribution function $F_{X_J}(y)$.

**20.2.2** *Time or age from a life table*

Simulating multinomial probabilities step-by-step for each category, utilizing the property that the conditional distribution of one multinomial category given the other category is binomial distribution.

**20.2.3** *Simulating from the (a, b, 0) class*

The process is in general, noting that the first event carries an index of 0, the second event an index of 1, and so on:

1. Simulate the time of the first event as an exponential variable with mean $1/\lambda_0$. Determine this time as

$$t_0 = -\ln(1-u_0)/\lambda_0,$$

where $u_0$ is a pseudo-uniform random number.

2. Let $t_{k-1}$ be the time of the most recently simulated event. Simulate the time to the next event using an exponential variable with mean $1/\lambda_k$. Determine this time as $s_k = -\ln(1-u_k)/\lambda_k$.

3. The time of the next event is then $t_k = t_{k-1} + s_k$.

4. Repeat steps 2 and 3 until $t_k > 1$.

5. The simulated value is $k$.

**20.2.4** *Normal and lognormal distributions*

A) Inversion Method

B) A simple alternative is the Box-Muller transformation.

1. The method begins with the generation of two-independent pseudouniform random numbers $u_1$ and $u_2$.
2. Then two independent standard normal values are obtained from

$$z_1 = \sqrt{-2\ln(u_1)}\cos(2\pi u_2) \; and \; z_1 = \sqrt{-2\ln(u_1)}\sin(2\pi u_2).$$

C) An improvement is the polar method, which also begins with two pseudouniform values. The steps are:

1. Calculate $x_1 = 2u_1 - 1$ and $x_2 = 2u_2 - 1$.
2. Calculate $w = x_1^2 + x_2^2$.
3. If $w \geq 1$, repeat steps 1 and 2. Else proceed to step 4.
4. Calculate $y = \sqrt{-2\ln(w)/w}$.
5. Calculate $z_1 = x_1 y$ and $z_2 = x_2 y$.

**20.3** *Determining the Sample Size*

Solve for $n$: $\dfrac{0.01\mu}{\sigma/\sqrt{n}} = 1.96, \quad (20.1) \quad \dfrac{0.01\mu}{\sigma/\sqrt{n}} = z_{\alpha/2}, \quad (20.1a) \quad \dfrac{0.01q}{\sqrt{q(1-q)}/\sqrt{n}} = z_{\alpha/2}, \quad (20.1b)$

**20.4.5** Statistical Analyses

Boostrap MSE: $MSE_{\widehat{\theta}}(\theta) = \sum_{all \text{ bootstrap data}} P(Data)(\widehat{\theta} - \theta)^2$