King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 690
Final Exam– 2014–2015 (141)
Tuesday, December 30, 2014
Allowed Time: 180 minutes

Instructor: **Dr. Boubaker Smii**

Name: ______________________________

ID #: ____________________

Section #: ___________ Serial Number: ___________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justification !

Question #	Grade	Maximum Points
1		08
2		08
3		10
4		10
5		14
6		20
7		20
8		20
Total:		70

Exercise 1:

Let $X_1, X_2, ...$ be a sequence of random variables adapted to a filtrations $\mathcal{F}_1, \mathcal{F}_2, ...$

1- Show that if X_n is a martingale with respect to $\mathcal{F}_n$, then

$$\mathbb{E}(X_1) = \mathbb{E}(X_2) = ... \tag{a}$$

2- Suppose that X_n is a martingale with respect to a filtration $\mathcal{F}_n$. Show that X_n is a martingale with respect to the filtration $\mathcal{G}_n = \sigma(X_1, ..., X_n)$.

Exercise 2:

Let ψ be a characteristic function of a given random variable X.

1- Prove that

$$Re\Big(1-\psi(t)\Big) \geq \frac{1}{4}\,Re\Big(1-\psi(2t)\Big) \tag{b}$$

2- Deduce from equation (b) that

$$1-\mid \psi(2t) \mid \leq 8\,\Big(1-\mid \psi(t) \mid\Big) \tag{c}$$

Exercise 3: Consider a Markov Chain consisting of the three states $0, 1, 2$ and having transition probability matrix:

$$\mathbf{P} = \begin{pmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \\ 0.6 & 0.1 & 0.3 \end{pmatrix}$$

1. Graph the given Markov Chain.
2. Compute the two step transition matrix $\mathbf{P}^2$.
3. Find $P(X_2 = 1 \mid X_1 = 0)$.

Exercise 4:

Let $\{X_t, t \geq 0\}$ be an Itô process, a, b real constants and B_t the standard Brownian motion.

1- Find the solution of the stochastic differential equation:

$$dX_t = (a + \frac{b^2}{2})\, X_t\, dt + b\, X_t\, dB_t, \qquad \text{(d)}$$

with initial condition $X(0) = X_0$.

2- Find the solution of the Linear stochastic differential equation

$$dX_t = a\, X_t\, dt + b\, X_t\, dB_t, \qquad \text{(e)}$$

with initial condition $X(0) = X_0$.

Exercise 5:

To describe the motion of a pendulum with small, random perturbations in its environment we consider the stochastic differential equation :

$$Y_t'' + \Big(1 + \epsilon W_t\Big) Y_t = 0; \qquad Y_0,\, Y_0' \; given, \tag{f}$$

where W_t is a one-dimensional white noise, ϵ a positive constant.

1- Show that the stochastic differential equation (f) can be written in the following form:

$$dX_t = K\, X_t\, dt - \epsilon\, L\, X_t\, dB_t, \tag{g}$$

where X_t, K, L are suitable matrices and B_t a Brownian motion.

2- Show that Y_t solves a stochastic Volterra equation of the form

$$Y_t = Y_0 + Y_0'\; t \; + \int_0^t a(t,r)\, Y_r\, dr \; + \int_0^t \gamma(t,r)\, Y_r\, dB_r\,, \tag{h}$$

where $a(t,r) = r - t$ and $\gamma(t,r) = \epsilon\,(r - t)$.

Exercise 6:

Consider a model where the stock prices follow an Ornstein-Uhlenbeck process $S(t)$.

1. Write down a stochastic differential equation for $S(t)$.

2. Solve the stochastic equation found in 1.

3. Find the probability that at a certain time $t_1 > 0$ we will have negative prices. (Express the result as $\Phi(x)$, where Φ is the standard Normal distribution function and x a given real number.)

Exercise 7:

Let $t_0^n < t_1^n < \cdots < t_n^n = T$, where $t_i^n = \frac{iT}{n}$, be a partition of the interval $[0, T]$ into n equal parts. We denote by

$$\Delta_i^n B = B(t_{i+1}^n) - B(t_i^n) \qquad \text{(i)}$$

the corresponding increments of the Brownian motion $B(t)$.

Show that

$$\lim_{n \longrightarrow +\infty} \sum_{i=0}^{n-1} (\Delta_i^n B)^2 = T, \quad in \;\; L^2. \qquad \text{(j)}$$

Exercise 8:

Let X_t be the price of a risky asset at time t. Assume that an investment of β_0 in bond yields an amount of $\beta_0\, e^{r\,t}$, $r > 0$.

1- Write down an expression of the portfolio V_t depending on X_t and β_t.

2- Assume that the value V_t of your portfolio at time t is given by:

$$V_t = u(T - t,\, X_t), \;\; t \in [0,\, T],$$

for some smooth deterministic function $u(t,\, x)$ and X_t a stochastic process satisfying:

$$X_t = X_0 + c \int_0^t X_s\, ds + \sigma \int_0^t X_s\, dB_s, \quad c > 0,\, \sigma > 0.$$

Express V_t in terms of u_1, u_2 and u_{22}.