King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 690
Exam I– 2014–2015 (141)
Monday, November 24, 2014
Allowed Time: 120 minutes

Instructor: **Dr. Boubaker Smii**

Name: ______________________________

ID #: ____________________

Section #: ___________ Serial Number: ___________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justification !

Question #	Grade	Maximum Points
1		08
2		08
3		10
4		10
5		14
6		20
Total:		70

Exercise 1:

Let $N(t)$, $t \geq 0$ be a Poisson process with parameter λ. Show that $\lim\limits_{t \longrightarrow \infty} \dfrac{N(t)}{t} = \lambda$ a.s.

Hint: You may use the Strong law of large numbers : $\lim\limits_{n \longrightarrow \infty} \dfrac{N(n)}{n} = \lambda$ a.s.

Exercise 2:

Consider the standard Brownian motion $\{B_t,\, t \geq 0\}$.

a)- Find $\mathbb{E}(\mid B_t - B_s \mid^2)$.

b)- Given that $\displaystyle\int_{\mathbb{R}} e^{\frac{-(x - i\,\lambda\, t)^2}{2t}}\, dx = \sqrt{2\,\pi}$, compute the characteristic function of B_t.

c)- Deduce from b) $\mathbb{E}(B_t^4)$.

Exercise 3:

Consider the geometric Brownian motion given by $X_t = e^{\mu t + \sigma B_t},\ t \geq 0,\ \sigma > 0,\ \mu \in \mathbb{R}$.

1- Find $\mathbb{E}(X_t)$ and $Var(X_t)$.

2- Give an application where we can use the geometric Brownian motion X_t.

<u>Exercise 4:</u>I- Let μ be a probability measure. Prove that $\hat{\mu}$ is a bounded, continuous function with $\hat{\mu}(0) = 1$.

II- Let X be a random variable exponentially distributed with parameter λ. Prove that $\mathbb{E}(X_t) = \frac{\lambda}{(\lambda - it)}$, $t \geq 0$.

Exercise 5: I- a). Give two major differences between the Riemann and Itô integrals.

II- Let X_t, Y_t be Itô processes in $\mathbb{R}$.

1)- Prove that:

$$X_t\,Y_t = X_0\,Y_0 + \int_0^t Y_s\,dX_s + \int_0^t X_s\,dY_s + \int_0^t dX_s\,dY_s$$

2)- Let $F_t = \exp(-\,\alpha\,B_t + \frac{1}{2}\,\alpha^2\,t),\ \alpha\,\in\,\mathbb{R}$.

i)- Find dF_t.

ii)- Given that: $dY_t = r\,dt + \alpha\,Y_t\,dB_t,\ r\,\in\,\mathbb{R}$. Prove that $Y_t = Y_0\,F_t^{-1}\,+\,r\,F_t^{-1}\int_0^t F_s\,ds$

(**Hint**: Use 1)).

Exercise 6:

Let $B_t \in \mathbb{R}$, $B_0 = 0$. Define $X_k(t) = \mathbb{E}(B_t^k), \;\; k = 0, 1, 2, ...; \; t \geq 0$.

1)- Prove that $X_k = \frac{k(k-1)}{2} \displaystyle\int_0^t X_{k-2}(s)\, ds, \;\; k \geq 2$.

2)- Deduce $\mathbb{E}(B_t^4)$ and $\mathbb{E}(B_t^6)$

3)- Show that $\mathbb{E}(B_t^{2k+1}) = 0$ and $\mathbb{E}(B_t^{2k}) = \frac{(2k)!t^k}{2^k\, k!}; \;\; k = 1, 2, ...$