King Fahd University of Petroleum and Minerals
Department of Mathematic

Math 580: Convex Analysis
Final Exam
Fall 2014
Time Limit: 180 Minutes

Student ID :
Student Name :

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
5	25	
6	25	
7	25	
8	25	
9	10	
10	10	
Total	220	

1. (25 points) Let C be a nonempty subset of $\Re^n$. For $x \in C$ and $y \in \Re^n$, define the set

$$\Lambda_{x,y} = \{\lambda \in \Re \mid x + \lambda y \in C\}.$$

Show that C is convex if and only if $\Lambda_{x,y}$ for all $x \in C$ and $y \in \Re^n$.

2. (25 points) Let C be a nonempty subset of $\Re^n$. Show that

   (a) $\text{ri}(C) = \text{ri}(\text{cl}(C))$.

   (b) $\text{cl}(C) = \text{cl}(\text{ri}(C))$.

3. (25 points) Left $f : \mathfrak{R}^n \to \overline{\mathfrak{R}}$ be a convex function with $\bar{x} \in \mathrm{dom}(f)$. Assume that f is bounded above on $\mathbb{B}_\delta(\bar{x})$ for some $\delta > 0$. Show that

   (a) f is bounded on $\mathbb{B}_\delta(\bar{x})$.

   (b) f is Lipschitz continuous on $\mathbb{B}_{\delta/2}(\bar{x})$.

4. (25 points) Show that the problem of finding a point $\bar{x} \in \Re^n$ such that

$$-\nabla f(\bar{x}) \in N(\bar{x}, \Omega)$$

is equivalent to the following convex optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & x \in \Omega \end{array}$$

where $f : \Re^n \to \Re$ is a convex and differentiable function and $\Omega \subseteq \Re^n$ is a convex set.

5. (25 points) Let $f : \Re^n \to \Re$ be a convex function. Show that for every $x \in \Re^n$, we have

$$f'(x; y) = \max_{d \in \partial f(x)} y'd.$$

6. (25 points) Consider the following function $f : \mathfrak{R} \to \mathfrak{R}$

$$f(x) = \max\{x^2, x^2 - 4x + 4\}.$$

   (a) Show that f is convex. (You may use graph of f to show its convexity)

   (b) Compute $\partial f(-1), \partial f(1)$ and $\partial f(2)$.

7. (25 points) Let $\Omega_1 = \big\{(x,y) \in \Re^2 \mid x \geq 1, y \geq 2\big\}$, $\Omega_2 = \big\{(x,y) \in \Re^2 \mid (x-1)^2 + (y-2) \leq 25\big\}$ and consider the following extended real-valued function $f : \Re^2 \to \overline{\Re}$

$$f(x) = \begin{cases} x^2 + y^2 - \sin(\pi x y), & (x,y) \in \Omega_1 \cap \Omega_2 \\ \\ \infty, & \text{otherwise.} \end{cases}$$

(a) Plot the effective domain of f.

(b) Using graphs only (one separate graph for each requirement), find $\partial^\infty f(1,2)$, $\partial^\infty f(2,2)$ and $\partial^\infty f(5,3)$.

(c) Find $(\bar{x}, \bar{y})$ such that $\partial f(\bar{x}, \bar{y}) = \{(3,6)\}$

8. (25 points) Let $f : \Re \to \overline{\Re}$ be the following function

$$f(x) = \begin{cases} -\ln(x), & x > 0, \\ \infty, & \text{otherwise.} \end{cases}$$

(a) Find the Fenchel conjugate for f.

(b) Find the Fenchel biconjugate for f at $x = e^3$.

9. (10 points) Let S be an arbitrary subset S of $\mathfrak{R}^n$. The negative polar cone of S, denoted by S^*, is defined as

$$S^* = \{y \in \mathfrak{R}^n \mid \langle y, x\rangle \leq 0 \text{ for all } x \in S\}.$$

Show that

(a) S^* is a cone.

(b) If $S_1 \subset S_2$ then $S_2^* \subset S_1^*$.

(c) $S_1^* \cup S_2^* \subset (S_1 \cap S_2)^*$.

10. (10 points) Let $\Omega \subseteq \Re^n$ be a convex set. A function $f : \Omega \to \Re$ is said to be *quasiconvex* on Ω if

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\} \qquad \text{for all} \quad x, y \in \Omega \text{ and all } \lambda \in [0,1].$$

Show that f is quasiconvex on Ω if and only if the set

$$L_\alpha(f) = \{x \in \Omega | f(x) \leq \alpha\}$$

is convex for every $\alpha \in \Re$.