King Fahd University of Petroleum and Minerals
Department of Mathematic

Math 580: Convex Analysis
Major Exam 1
Fall 2014
Time Limit: 120 Minutes

Student ID :
Student Name :

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	20	
3	20	
4	20	
5	20	
6	10	
Total	100	

1. (10 points) Show that the set C, defined as

$$C := \left\{(x_1, x_2, x_3) \in \mathfrak{R}^3 : 1 - x_1^2 - x_2^2 - x_3^2 \geq 0\right\},$$

is convex.

2. (20 points) Let K be a nonempty convex cone in $\Re^n$. Prove that $K = -K$ if and only if K is a linear subspace of $\Re^n$.

3. (20 points) Let ψ be a function defined on the interval $(a,b) \subset \Re$; that is $\psi : (a,b) \to \Re$.
Prove that ψ is convex on (a,b) if and only if

$$\psi(x_2) \le \frac{x_3 - x_2}{x_3 - x_1}\psi(x_1) \quad + \quad \frac{x_2 - x_1}{x_3 - x_1}\psi(x_3), \qquad \text{for all } x_1, x_2, x_3 \text{ such that } a < x_1 < x_2 < x_3 < b.$$

4. (20 points) Let C be a convex subset of $\Re^n$ and $f : C \to (-\infty, \infty)$ be a function defined on C. Define the function $\phi : [0, 1] \to (-\infty, \infty)$ as

$$\phi(t) = f(tx + (1-t)y), \quad \text{for all} \quad x, y \in [0, 1].$$

Prove that

(a) the function f is convex on C if and only if the function ϕ is convex on $[0, 1]$,

(b) the derivative of the function ϕ is nondecreasing.

5. (20 points) Let C and D be two nonempty convex subsets of $\Re^n$.
   (a) If D is closed and $C \subset D$, prove that the recession cone of D contains the recession cone of C.
   (b) Prove that the recession cones of $\mathrm{ri}(C)$ and $\mathrm{cl}(C)$ coincide.

6. (10 points) Consider the following optimization problem

$$\begin{array}{cc} \min & f(x) \\ \text{subject to} & \\ & x \in X \end{array} \tag{P}$$

where $X \subset \Re^n$ and f is real-valued and differentiable function defined on an open set containing X. If $x^* \in X$ is a minimum of f, show that $y'\nabla f(x^*) \geq 0$ for every $y \in \Re^n$ such that $x^* + \alpha y \in X$ for all $\alpha \in [0, \bar{\alpha}]$ for some $\bar{\alpha} > 0$.