King Fahd University of Petroleum and Minerals,

Department of Mathematics and Statistics- Term 141

Final Exam : Math 550, Linear Algebra

Duration: 3 Hours

NAME :

ID :

Exercise 1. (5-5-5)

Let $V = \mathbb{R}^3$ and let $B = \{(1,0,1),(1,1,1),(0,0,1)\}$ and $B' = \{(1,0,1),(0,1,0),(-1,0,0)\}$ two bases for V.

(1) Find the transition matrix P from B' to B.

(2) Let T a linear operator on V with $[T]_B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix}$. Find $[T]_{B'}$.

(3) Is T an isomorphism?

Exercise 2. (5-5-5-5 points)

Let V be an m-dimensional vector space over F and T a linear operator with m distinct characteristic values $c_1 = 1, c_2, \ldots, c_m$ such that $|c_j| < 1$ for $j = 2, \ldots n$.

(1) Prove that for every vector $\alpha \in V$, $lim\ T^n\alpha$ exists.

We define a linear operator U on V by $U\alpha = lim\ T^n\alpha$.

(2) Find $dim(Nullspace(U))$ and a basis B_1 for it.

(3) Find $dim(range(U))$ and a basis B_2 for it.

(4) Find the matrix representing U in the basis $B = B_1 \cup B_2$.

Exercise 3. (4-5-5-6)

Let V be an n-dimensional vector space over the field of rational numbers $\mathbb{Q}$ and T a non-singular linear operator on V such that $T^{-1} = T^2 + T$.

(1) Find the minimal polynomial of T.

(2) Prove that 3 divides $dim V$.

(3) Find the rational form matrix associated to T.

(4) Prove that every nonzero vector in V is a cyclic vector if and only if $dim V = 3$.

Exercise 4. (4-4-6-6 points)

Let V be an n-dimensional vector space over a field F and T a linear operator.

(1) Assume that T has exactly n distinct characteristic values:

(i) Prove that every linear operator U commutating with T is diagonalizable.

(ii) Prove that if U is a nilpotent operator commutating with T, then $U = 0$.

(2) Conversely, assume that T commutes with no nonzero nilpotent operator.

(iii) Prove that if c is a multiple characteristic value of T, then the Jordan block matrix J_c in the Jordan matrix form of T commutes with a nilpotent matrix B_c to be determined.

(iv) Use the result of (iii) to prove that T has exactly n characteristic values.

Exercise 5. (5-5-5-5)

Let $V = \mathbb{R}^3$ endowed with its standard inner product and let B the ordered basis $B = \{u_1 = (1, 0, 1), u_2 = (1, 1, 1), u_3 = (0, 0, 2)\}$.

(1) Apply Gram-Schmidt process to B to obtain an orthonormal ordered basis $B' = \{v_1, v_2, v_3\}$ for V.

(2) Let $W = span\{v_2, v_3\}$ and E the orthogonal projection of V onto W. Find a formula for $E(x, y, z)$.

(3) Find $[E]_S$, the matrix representing E is the standard basis S.

(4) Find $[E]_{B'}$.

Exercise 6. (5-5-5-5-5)

Let V be an n-dimensional complex inner product space and T a normal linear operator.

(1) Prove that $Nullspace(T) = Nullspace(T^*)$.

(2) Prove that $Nullspace(T)$ is the orthogonal complement of $range(T)$ (that is, $Nullspace(T) = (rang(T))^{\perp}$. Deduce that $Nullspace(T) = Nullspace(T^2)$.

(3) Suppose that there is two polynomials $f(X)$ and $g(X)$ relatively prime and $\alpha, \beta \in V$ such that $f(T)\alpha = g(T)\beta = 0$. Prove that $(\alpha|\beta) = 0$.

(4) Prove that there exist two linear self-adjoint operators T_1 and T_2 with $T_1T_2 = T_2T_1$ such that $T = T_1 + iT_2$ (i is the complex number with $i^2 = -1$).

(5) Prove that there is a polynomial $h \in \mathbb{C}[X]$ such that $T^* = h(T)$.

Exercise 7. (5-5-5-5)

Let $V = \mathbb{R}^3$, $S = \{e_1, e_2, e_3\}$ its standard basis and f the skew symmetric bilinear form on V defined by $f(X, Y) = x_1y_2 - x_1y_3 - x_2y_1 + 2x_2y_3 + x_3y_1 - 2x_3y_2$.

(1) Find $[f]_S$.

(2) Find $rank(f)$.

(3) Let $W = span\{e_1, e_2\}$. Find a basis B for $W^{\perp}$.

(4) Find $[f]_{B'}$ where B' is the basis $B' = \{e_1, e_2\} \cup B$.