King Fahd University of Petroleum and Minerals,

Department of Mathematics and Statistics- Term 141

Major Exam 1: Math 550, Linear Algebra

Duration: 2 Hours

NAME :

ID :

Solve the following Exercises.

Exercise 1. (5-5-5-5 points)

(1) Prove that every vector space V over a field F has a basis.

(2) Let W be a subspace of V and S_0 a basis of W. Prove that V has a basis S containing S_0. (Notice that V is not necessarily of finite dimensions)

(3) Prove that every subspace W of a vector space V has a complement, that is, there is a subspace U of V such that every element $x \in V$ can be expressed in a unique way as $x = a + b$ where $a \in W$ and $b \in U$.

(4) Let V be the vector space of all real-valued functions and W its subspace of all odd functions. Find a complement of W.

Exercise 2. (7-6-7)

(1) Find explicitly a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such that $T(1,1,0) = (1,0)$ and $T(1,0,1) = (0,1)$.

(2) Find the matrix representing T in the standard bases $\mathcal{S}_1, \mathcal{S}_2$ of $\mathbb{R}^3$ and $\mathbb{R}^2$.

(3) Let $\mathcal{B}_1 = \{(1,1,0),(1,0,1),(0,0,1)\}$ and $\mathcal{B}_2 = \{(1,1),(1,2)\}$. Find the matrix representing T in the bases $\mathcal{B}_1$ and $\mathcal{B}_2$.

Exercise 3. (4-4-6-6 points)

Let V be a vector space over a field F and T a linear operator on V.

(1) Prove that $ker(T) \subseteq ker(T^2)$ and $range(T^2) \subseteq range(T)$.

(2) Prove that if $T^2 = 0$, then $range(T) \subseteq ker(T)$.

(3) Assume that V is of finite dimension and $range(T^2) = range(T)$. Prove that $ker(T) \cap rang(T) = \{0\}$.

(4) Find an example of an infinite dimensional vector space V with a linear operator T such that $range(T^2) = range(T)$ but $ker(T) \cap rang(T) \neq \{0\}$.

Exercise 4. (5-5-5-5)

Let V be an n-dimensional vector space over a field F and T and S two linear operators on V.

(1) Suppose that there are (ordered) bases B_1 and B_2 of V such that $[T]_{B_1} = [S]_{B_2}$. Prove that there is an invertible operator U on V such that $S = UTU^{-1}$.

(2) Conversely, suppose that there is an inverse operator U on V such that $S = UTU^{-1}$. Prove that there are bases B_1 and B_2 of V such that $[T]_{B_1} = [S]_{B_2}$.

(3) Application: Set $V = \mathbb{R}^2$, T, S defined by $T(a, b) = (2a, 3b)$ and $S(a, b) = (3a, 2b)$.

(i) Find two ordered bases B_1 and B_2 of V such that $[T]_{B_1} = [S]_{B_2}$.

(ii) Find an inverse operator U on V such that $S = UTU^{-1}$.

Exercise 5. (5-5-5-5)

Let F be a field and V be a finite dimensional space over F.

(1) Find all linear transformations of F as a vector space over itself.

(2) Let f a linear functional on V such that $W = ker f$ is a hypersubspace of V. Prove that for every linear functional g of V such that $g(W) = 0$, there is a scalar c such that $g = cf$.

(3) Let T be a linear operator on V and d a scalar in F such that $Tu = du$ for some non-zero vector $u \in V$. Prove that there is a non-zero linear functional h on V such that $T^t(h) = dh$.

(4) Assume that $F = \mathbb{R}$, $V = \mathbb{R}^2$ and T is defined by $T(a, b) = (2a, o)$. Find u, d and h satisfying question (3).