King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH 535 [Functional Analysis I]
Semester (141)

Final Exam: January 04, 2015 **Time allowed: 3hrs**

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(Q1) (a) Define a convex set in a normed space X. Show that the closed unit ball of X is convex.

(b) Define norm of a linear transformation from a normed space X into another normed space Y. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is defined by $f(x) = x_1 + x_2 + x_3$ where $x = (x_1, x_2, x_3)$ and $\| x \| = \left(\sum_{i=1}^{3} |x_i|^2 \right)^{1/2}$. Show that $\| f \| = \sqrt{3}$.

(Q2) (a) Show that any two norms on a finite dimensional normed space are equivalent.

(b) Let f be a bounded linear functional on a subspace Z of a normed space X. Prove that there exists a bounded linear functional $\tilde{f}$ on X which is an extension of f on X and $\| f \|_Z = \| \tilde{f} \|_X$.

(Q3) (a) Let $\{T_n\}$ be a sequence of bounded linear transformations from a Banach space X into a normed space Y such that $\| T_n x \|$ is bounded for every $x \in X$. Then prove that the sequence $\{\| T_n \|\}$ is bounded.

(b) Let A be a subset of a normed space X. Use part (a) to show that if $f(A)$ is bounded for all $f \in X^*$, then A is bounded.

(Q4) (a) Let $(X, \| \cdot \|)$ be a real normed space. If the law of parallelogram holds in X, then prove that X is an inner product space.

(b) Let C_{00} be the space of all real sequences with all terms zero after a finite number of terms. Define inner product on C_{00} by

$< x, y >= \sum_{n=1}^{\infty} x_n y_n$ for $x = \{x_n\}, y = \{y_n\}$ in C_{00}.

Consider the subspace $S = \left\{ x \in C_{00} : \sum_{n=1}^{\infty} \frac{x_n}{2^{n/2}} = 0 \right\}$

and $f : C_{00} \to (R, |\cdot|)$ defined by $f(y) = \sum_{n=1}^{\infty} \frac{y_n}{2^{n/2}}$. Show that f is continuous and S is a closed subspace of C_{00}.

Why $C_{00} = S \oplus S^+$ is not valid?

(Q5) (a) Let f be a continuous linear functional on a Hilbert space H. Prove that there exists a unique $z \in H$ such that $f(x) =< x, z >$ for all $x \in H$ and $||f|| = ||z||$.

(b) Use part (a) to show that any Hilbert space is reflexive.

(Q6) (a) Introduce strong topology Γ_s and weak Topology Γ_w on a normed space X and verify $\Gamma_w \subset \Gamma_s$. Is it true that weakly convergent sequences are strongly convergent. Justify your answer.

(b) Define a complex Banach algebra A with identity e. Prove that the set of all non-invertible elements of A is a closed set.