King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
First Semester 2014-15(141)
MATH 535[Functional Analysis I]

Exam 1: November 03, 2014 Time allowed: 2hrs.

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(Q1) (a)If $1 \leq p < \infty$ and $\{\xi_j\}$ and $\{\eta_j\}$are sequences of numbers, then verify that

$$\left(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^{\infty} |\xi_j|^p\right)^{1/p} + \left(\sum_{j=1}^{\infty} |\eta_j|^p\right)^{1/p}.$$

(b) Let p be a fixed integer such that $1 \leq p \leq \infty$. Define $\ell_p = \{x = \{\alpha_n\} : \sum_{n=1}^{\infty} |\alpha_n|^p < \infty\}$. Prove that the space ℓ_p is a Banach space under the norm $\| x \| = (\sum_{n=1}^{\infty} |\alpha_n|^p)^{1/p}$.

(Q2) (a) Every metric on a vector space is not necessarily a norm. Justify this statement by means of a suitable example.

(b) Prove that every finite dimensional subspace Y of a normed space X is complete.

(Q3) (a) Suppose that $f : C[a,b] \to \mathbb{R}$ is defined by $f(x) = \displaystyle\int_a^b x(t)dt$ where $\| x \| = \max_{t\in[a,b]} |x(t)|$. Calculate $\| f \|$.

(b) Use appropriate results (do not prove them) to find C_0^{**} where C_0 is the vector space of sequences of numbers converging to 0 and is equipped with sup. norm.

(Q4) (a) State Banach fixed point theorem. Use this result to find a unique fixed point of $T : [1,\infty) \to [1,\infty)$ defined by $Tx = \dfrac{25}{26}\left(x + \dfrac{1}{x}\right)$.

(b) Determine whether the norms $\displaystyle\sum_{n=1}^{\infty} |x_n|$ and $\displaystyle\sup_{1\leq n\leq\infty} |x_n|$ on the usual space ℓ_1, are equivalent.