King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH533 - Complex Variables
Final Exam – Semester I, 2014-2015

Exercise 1

Let $a, b \in \mathbb{C}$ such that Re $a > 0$, Re $b > 0$ and $a \neq b$. Evaluate

$$\int_{-\infty}^{+\infty} \frac{x^3 e^{ix}}{(x^2 + a^2)(x^2 + b^2)} \, dx$$

Exercise 2

Show that the equation $z^5 + 15z + 1 = 0$ has precisely four solutions in the annulus $A = \{z \in \mathbb{C} : 3/2 < |z| < 2\}$.

Exercise 3

Let

$$f(z) = \frac{z^3 \sin(\frac{\pi}{z})}{(z-1)^2} \text{ in } \mathbb{C}.$$

(a) Identify the singularities of f.

(b) Classify each singularity as a removable, a pole (order), or an essential singularity.

(c) Find $\oint_{|z|=2} f(z)dz$.

Exercise 4

Use the Cauchy integral formula to evaluate the following integrals.

(a) $\oint_{|z|=1} (\text{Re}z)^2 \, dz$

(b) $\oint_{|z-1|=1} (\overline{z})^2 \, dz$

Exercise 5

(a) Show that $\displaystyle\int_{|w|=r} \frac{dw}{w^n(w-z)^2} = 0$, for $|z| < r$ and $n \geq 0$.

Let f be analytic function on $|z| \leq r$.

(b) Deduce from (a) that $\displaystyle\int_{|w|=r} \frac{\overline{f}(w)}{(w-z)^2}\, dw = 0$, for $|z| < r$.
(Hint: Use the Taylor expansion of f at 0.)

(c) Find $\displaystyle\int_{|w|=r} \frac{\text{Re } f(w)}{(w-z)^2}\, dw$, for $|z| < r$.

(d) Find $\displaystyle\int_{|w|=r} \frac{\text{Re } f(w)}{w-z}\, dw$, for $|z| < r$. (Hint: compute $\displaystyle\int_{|w|=r} \frac{dw}{w^n(w-z)}$ for $|z| < r$ and $n \geq 0$.)

Exercise 6

Let $\omega : [0, 1] \to \mathbb{R}$ be of class $\mathcal{C}^\infty$. Define

$$f(z) = \int_0^1 t^{z-1}\omega(t)\, dt$$

(a) Show that f is analytic in $\operatorname{Re} z > 0$.

(b) Find f, for $\omega(t) = a_0 + a_1 t + \ldots + a_n t^n$. Show that f has a meromorphic extension to $\mathbb{C}$ and compute the resides at the poles of f.

Assume that $\omega(1) = 0$.

(c) Show that f has a meromorphic extension to all $\mathbb{C}$ with simple poles at $z = 0, -1, -2, ...$ (Hint: Use integration by parts to show that $f(z) = \frac{f_1(z)}{z}$).

(d) Find $\operatorname{Res}(f, 0)$ and $\operatorname{Res}(f, -1)$.

Exercise 7

(a) Suppose that f is an *entire* function such that $|f(z)| \leq |Q(z)|$, for $|z|$ large, where Q is a polynomial. Show that f is a polynomial.

(b) Suppose that f is analytic in the complex plane with an exception of a pole of order 2 at $z = 0$ and a simple pole at $z = 1$. Suppose further that

$$|f(z)| \leq |z|^4 \text{ for all } z \text{ with } |z| \geq 2.$$

Show that $f(z) = \dfrac{P(z)}{z^2(z-1)}$, where P is a polynomial, and find the degree of P.

(c) Suppose that g is a meromorphic function over $\mathbb{C}$ such that

$$|g(z)| \leq 1 \text{ for } |z| \geq 2014.$$

Show that g has a finite number of poles and deduce that g is a rational function.