King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH533 - Complex Variables
Midterm Exam – Semester I, 2014-2015

Exercise 1

Suppose that f is analytic for $|z| < 2$ and α is a complex constant. Evaluate

$$I = \int_{|z|=1} (\text{Re } z + \alpha) \frac{f(z)}{z} \, dz$$

Exercise 2

Let f be an entire function and $n \in \mathbb{N}$, $n \geq 1$.

(i) Find $I_f = \displaystyle\int_{|z|=2} \frac{f(z)}{(z-1)^{n+2}}\, dz$.

(ii) What is the value of I_f if f is a polynomial of degree n?

Exercise 3

True or false (if true, give a short explanation, if false, give a counterexample)

(i) If f is an entire function and bounded on a half-plane, then f is constant.

(ii) If f is analytic and bounded on $|z| > 1$, then f is constant.

(iii) If f is entire and bounded on $|z| > 1$, then f is constant.

(iv) If f is analytic in the punctured complex plane $\mathbb{C} \setminus \{0\}$ such that $f(1/n) = 0$, for all $n \geq 1$. Then $f = 0$.

(v) Suppose f is analytic in the annulus $1 \leq |z| \leq R$, $|f(z)| \leq R^n$ for $|z| = R$ and $|f(z)| \leq 1$ on $|z| = 1$. Then $|f(z)| \leq |z|^n$ in the annulus.

Exercise 4

In each case, exhibit a nonconstant f having the desired properties or explain why no such function exists:

(i) f is analytic in $|z| < 1$ with $f(\frac{1}{n}) = \frac{1}{n^2+1}$ for $n \in \mathbb{N}$.

(ii) f is analytic in $|z| < 1$ with $f(\frac{1}{n}) = \frac{(-1)^n}{n}$, $n \geq 1$

(iii) f is analytic in $|z| < 1$ with $f(\frac{1}{n}) = \frac{1}{\sqrt{n}}$, $n \geq 1$.

(iv) f is analytic in $\mathbb{C} \setminus \{0\}$ with $f'(z) = \frac{\cos z}{z}$

Exercise 5

Prove the following sharper version of the Schwarz's lemma: If $f : \Delta \to \Delta$ is analytic with $f(0) = f'(0) = ...f^{(n-1)}(0) = 0$, $n \in \mathbb{N}$, $n \geq 1$, then

$$|f(z)| \leq |z|^n \text{ for all } z \in \Delta \text{ and } |f^{(n)}(0)| \leq n!$$

Moreover, $f(z) = az^n$ for some $a, |a| = 1$ if and only if either $|f^{(n)}(0)| = n!$ or $|f(c)| = |c|^n$ for some $c \in \Delta \setminus \{0\}$.

Exercise 6

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ be the upper half-plane.

1. Show that $\varphi : \mathbb{H} \to \Delta$, $z \mapsto \dfrac{z-i}{z+i}$ is 1-1 analytic mapping and find φ^{-1}.
2. Let $f : \Delta \to \mathbb{H}$ be analytic, with $f(0) = i$. Show that
   (a) $\dfrac{1-|z|}{1+|z} \leq |f(z)| \leq \dfrac{1+|z|}{1-|z}$
   (b) $|f'(0)| \leq 2$.
3. Find all analytic functions $f : \Delta \to \mathbb{H}$, such that $f(0) = i$ and $|f'(0)| = 2$.