KFUPM, DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 421: FINAL EXAM, SEMESTER (141), DECEMBER 30, 2014

Math421: Introduction to Topology

Name : ....

ID : ....

Exercise	**Points**
1	____ / 20
2	____ / 20
3	____ / 20
4	____ / 20
5	____ / 20
6	____ / 20
7	____ / 24
8	____ / 20
9	____ / 11
Total	____ / 175

Exercise 1. We equip $\mathbb{R}^n$ with the usual topology. Let A be the closed unit ball of $\mathbb{R}^n$. Show that the interior $\mathrm{int}(A)$ of A is the open unit ball and the frontier $\mathrm{Fr}(A)$ is the unit sphere of $\mathbb{R}^n$.

Exercise 2. Let (X, d) be a a metric space. We equip $X \times X$ with the distance δ defined by: for all $(x, y), (a, b) \in E \times E$;

$$\delta((x, y), (a, b)) = d(x, a) + d(y, b).$$

(1) Show that for each $x, y, a, b \in X$, we have

$$|d(x, y) - d(a, b)| \leq d(x, a) + d(y, b).$$

(2) Deduce from (1) that the function

$$\begin{array}{rccc} d: & (X \times X, \delta) & \longrightarrow & (\mathbb{R}, |. - .|) \\ & (x, y) & \longmapsto & d(x, y) \end{array}$$

is uniformly continuous.

Exercise 3. Show that the function

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & x^2 \end{array}$$

is not uniformly continuous but its restriction to any closed bounded subset of $\mathbb{R}$ is uniformly continuous.

Exercise 4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function such that $|f'(x)| < \frac{1}{2}$, for all $x \in \mathbb{R}$. Show that f has a unique fixed point (use Banach fixed point Theorem).

Exercise 5. Let (X, d) be a metric space and (x_n) be a Cauchy sequence of X.

(a) Show that (x_n) is convergent if and only if it has a convergent subsequence.

(b) Suppose that (x_n) is nonconvergent. Show that $\{x_n : n \in \mathbb{N}\}$ is a closed set of X.

Exercise 6. Let $(X, d_1), (Y, d_2)$ be metric spaces and $f : X \longrightarrow Y$ be a uniformly continuous function. Show that if (x_n) is a Cauchy sequence of X, then $(f(x_n))$ is a Cauchy sequence of Y.

Exercise 7. Let $(X, \mathcal{T})$ be a topological space.

(a) Show that X is a T_1-space if and only if every singleton of X is closed.

(b) Suppose that $\mathcal{T} = \mathcal{CF}$ is the co-finite topology on X. Show that $(X, \mathcal{T})$ is is a T_1-space.

(c) Show that $(X, \mathcal{CF})$ is a T_2-space if and only if X is finite.

(d) Let R be an equivalence relation on X and X/R be the quotient space. Show that X/R is a T_1-space if and only if every equivalence class of R is closed in X.

Exercise 8. We equip $\mathbb{R}^n$ with the usual topology. Give a homeomorphism between the one-point compactification of $\mathbb{R}^n$ and the sphere

$$S^n := \{u \in \mathbb{R}^{n+1} : \|u\| = 1\}.$$

Exercise 9. Let $a < b$ be in $\mathbb{R}$. Show that there is no homeomorphism between the two intervals $[a, b[$ and $]a, b[$.