KFUPM, DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 421: EXAM I, SEMESTER (141), OCTOBER 23, 2014

Math421: Introduction to Topology

Name : ....

ID : ....

Exercise	**Points**
1	____ / 50
2	____ / 25
3	____ / 25
Total	____ / 100

Exercise 1. Let $\mathcal{U}$ be the usual topology on $\mathbb{R}$. Consider the two topologies on $\mathbb{R}$ defined by:

$$\mathcal{LL} = \{V \subseteq \mathbb{R} : \text{ if } x \in V, \text{ then there exist } a, b \in \mathbb{R} \text{ such that } x \in [a, b[\subseteq V\}$$

and

$$\mathcal{UL} = \{V \subseteq \mathbb{R} : \text{ if } x \in V, \text{ then there exist } a, b \in V \text{such that } x \in]a, b] \subseteq V\}.$$

The topology $\mathcal{LL}$ (resp. $\mathcal{UL}$) is called the *lower limit topology* (resp., *upper limit topology*) on $\mathbb{R}$.

(1) Show that $\mathcal{B}_1 = \{[a, b[: a, b \in \mathbb{R}\}$ is a basis of $\mathcal{LL}$ and $\mathcal{B}_2 = \{]a, b] : a, b \in \mathbb{R}\}$ is a basis of $\mathcal{UL}$.

(2) Show that:
- $\mathcal{U} \leq \mathcal{LL}$ and $\mathcal{U} \neq \mathcal{LL}$
- $\mathcal{U} \leq \mathcal{UL}$, and $\mathcal{U} \neq \mathcal{UL}$
- $\mathcal{LL} \nleq \mathcal{UL}$ and $\mathcal{UL} \nleq \mathcal{LL}$.

(3) Show that the function

$$\begin{array}{rcl} f \quad : \quad (\mathbb{R}, \mathcal{LL}) & \longrightarrow & (\mathbb{R}, \mathcal{UL}) \\ x & \longmapsto & -x \end{array}$$

is a homeomorphism.

Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$g(x) = \begin{cases} x+1 & \text{if} \quad x > 1 \\ \\ x & if \quad x \leq 1 \end{cases}$$

(4) Is

$$g \quad : \quad (\mathbb{R}, \mathcal{U}) \quad \longrightarrow \quad (\mathbb{R}, \mathcal{U})$$

continuous?

(5) Is

$$g \quad : \quad (\mathbb{R}, \mathcal{LL}) \quad \longrightarrow \quad (\mathbb{R}, \mathcal{U})$$

continuous?

(6) Is
$$g \;:\; (\mathbb{R},\mathcal{UL}) \;\longrightarrow\; (\mathbb{R},\mathcal{U})$$
continuous?

Exercise 2. Let X be a set and

$$\mathcal{T}_{cc} = \{U \subseteq X : U = \emptyset \text{ or } X - U \text{ is a countable set}\}.$$

(a) Show that $\mathcal{T}_{cc}$ is a topology on X.

(b) Show that $(X, \mathcal{T}_{cc})$ is discrete if and only if X is countable.

Exercise 3. Let X be a nonempty set and $d \in X$. Consider

$$\mathcal{T} = \{U \subseteq X : U = X \text{ or } d \notin U\}.$$

(a) Show that $\mathcal{T}$ is a topology on X.

(b) Show that for each $A \subseteq X$, we have $\overline{A} = A \cup \{d\}$.

(c) Suppose that $d \in A$; then find $\text{int}(A)$.

(d) Suppose that $X = \{a, b, c, d\}$. List all the open sets and closed sets of $(X, \mathcal{T})$.