

**King Fahd University of Petroleum and Minerals**  
**Department of Mathematics and Statistics**

Math 302 Final Exam

Semester (141) Dec. 30, 2014 at 8:00-11:00 AM

Name: .....

I.D: ..... Section: ..... Serial: .....

Question	Points
1	/14
2	/14
3	/14
4	/14
5	/14
6	/14
7	/14
8	/14
9	/14
10	/14
<b>Total</b>	<b>/140</b>



Question 1

(14 points)

a. Let  $A = \begin{pmatrix} -1 & 3 & 0 \\ 3 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ , use the Gaussian elimination method to find  $A^{-1}$ .

b. Use (a) to solve the system

$$\begin{cases} -x_1 + 3x_2 = 1 \\ 3x_1 - 2x_2 + x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases}$$

Sol.

a.  $\tilde{A} = \left[ \begin{array}{ccc|ccc} -1 & 3 & 0 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{-R_1 \\ R_2+3R_1}}$

$$\left[ \begin{array}{ccc|ccc} 1 & -3 & 0 & -1 & 0 & 0 \\ 0 & 7 & 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[ \begin{array}{ccc|ccc} 1 & -3 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 7 & 1 & 3 & 1 & 0 \end{array} \right] \xrightarrow{\substack{R_1+3R_2 \\ R_3-7R_2}}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 6 & -1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & -13 & 3 & 1 & -7 \end{array} \right] \xrightarrow{R_3/-13}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 6 & -1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3/13 & -1/13 & 7/13 \end{array} \right] \xrightarrow{\substack{R_1-6R_3 \\ R_2-2R_3}}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 5/13 & 6/13 & -3/13 \\ 0 & 1 & 0 & 6/13 & 2/13 & -1/13 \\ 0 & 0 & 1 & -3/13 & -1/13 & 7/13 \end{array} \right]$$

So  $A^{-1} = \begin{bmatrix} 5/13 & 6/13 & -3/13 \\ 6/13 & 2/13 & -1/13 \\ -3/13 & -1/13 & 7/13 \end{bmatrix}$

b)

b) we have  $AX = b$ .

$$\text{So } X = A^{-1}b$$

$$= \begin{bmatrix} \frac{5}{13} & \frac{6}{13} & -\frac{3}{13} \\ \frac{6}{13} & \frac{2}{13} & -\frac{1}{13} \\ -\frac{3}{13} & -\frac{1}{13} & \frac{7}{13} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{13} \\ \frac{6}{13} \\ -\frac{3}{13} \end{bmatrix}.$$

Question 2

(14 points)

Let  $A = \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix}$

a) Find all **eigenvalues** of  $A$  and the corresponding **eigenvectors**.

b) Find a matrix  $P$  that diagonalizes  $A$ .

c) If  $D = P^{-1}AP$ , find  $D^4$  and write a **formula** to evaluate  $A^4$ . (Do not calculate  $A^4$ )

Sol. a.

$$\begin{vmatrix} 1-\lambda & 0 & 7 \\ 0 & 1-\lambda & 0 \\ 7 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} + 7 \begin{vmatrix} 0 & 1-\lambda \\ 7 & 0 \end{vmatrix}$$

$$= (1-\lambda)^3 - 49(1-\lambda)$$

$$= (1-\lambda) [(\lambda-1)^2 - 49]$$

$$= (1-\lambda) [(\lambda-8)(\lambda+6)] = 0$$

$\Rightarrow \lambda = 1, -6, 8$  are eigenvalues.

Eigenvectors

$\lambda = -6$   $\begin{pmatrix} 7 & 0 & 7 \\ 0 & 7 & 0 \\ 7 & 0 & 7 \end{pmatrix} \xrightarrow[\begin{smallmatrix} R_2/7 \\ R_3-R_1 \end{smallmatrix}]{R_1/7} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow x_1 = -x_3, x_2 = 0$

So  $E_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$  is an eigenvector.

$\lambda = 1$   $\begin{pmatrix} 0 & 0 & 7 \\ 0 & 0 & 0 \\ 7 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 7 & 0 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow x_1 = x_3 = 0 \Rightarrow E_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$  is an eigenvector.

$\lambda = 8$   $\begin{pmatrix} -7 & 0 & 7 \\ 0 & -7 & 0 \\ 7 & 0 & -7 \end{pmatrix} \xrightarrow[\begin{smallmatrix} R_3+R_1 \\ R_2/-7 \end{smallmatrix}]{R_1/-7} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow x_2 = 0, x_1 = x_3 \Rightarrow E_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$  is an eigenvector

b.  $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$  diagonalizes  $A$  such that

$$P^{-1}AP = D = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

c.  $D^4 = \begin{pmatrix} 6^4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8^4 \end{pmatrix}$

$$A^4 = P D^4 P^{-1}$$

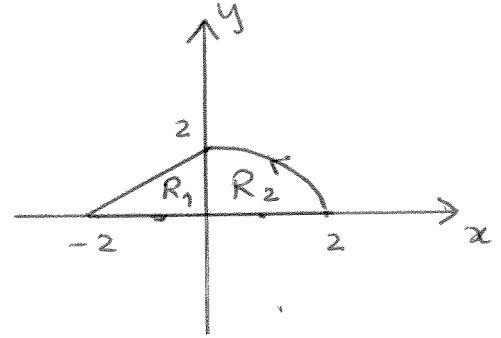
Question 3

(14 points)

Find  $\oint_C (-xy)dx + (e^{-y^2})dy$ , where  $C$  is the positively oriented closed path consisting of the quarter circle joining the points  $(2,0)$ ,  $(0,2)$ , the line segment joining the points  $(0,2)$ ,  $(-2,0)$ , and the line segment joining the points  $(-2,0)$ ,  $(2,0)$ .

sol.

$$\oint_C (-xy)dx + e^{-y^2} dy$$



$$= \iint_R \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \iint_R x dA$$

$$= \iint_{R_1} x dA + \iint_{R_2} x dA$$

$$= \int_{-2}^0 \int_0^{x+2} x dy dx + \int_0^{\pi/2} \int_0^2 (r \cos \theta) r dr d\theta$$

$$= \int_{-2}^0 x(x+2) dx + \int_0^{\pi/2} \cos \theta d\theta \int_0^2 r^2 dr$$

$$= \left[ \frac{x^3}{3} + x^2 \right]_{-2}^0 + \sin \theta \Big|_0^{\pi/2} \left[ \frac{r^3}{3} \right]_0^2$$

$$= \frac{8}{3} - 4 + (1) \left( \frac{8}{3} \right) = \frac{16}{3} - \frac{12}{3} = \frac{4}{3}$$

$$= \iint_{D_1} (x^2 + y^2)(4 - x^2 - y^2) dA - \iint_{D_2} (x^2 + y^2)(1 - x^2 - y^2) dA$$

We then use polar coordinates:

$$\text{flux} = \int_0^{2\pi} \int_0^2 r^2(4 - r^2) r dr d\theta - \int_0^{2\pi} \int_0^1 r^2(1 - r^2) r dr d\theta$$

$$= 2\pi \int_0^2 (4r^3 - r^5) dr - 2\pi \int_0^1 (r^3 - r^5) dr$$

~~we then use polar coordinates:~~

$$= 2\pi \left( r^4 - \frac{r^6}{6} \right) \Big|_0^2 - 2\pi \left[ \frac{r^4}{4} - \frac{r^6}{6} \right]_0^1$$

$$= 2\pi \left( 16 - \frac{64}{6} \right) - 2\pi \left( \frac{1}{4} - \frac{1}{6} \right)$$

$$= 2\pi \frac{32}{6} - \frac{2\pi}{12} = \frac{32\pi}{3} - \frac{\pi}{6} = \frac{21\pi}{2}$$



**Question 4**

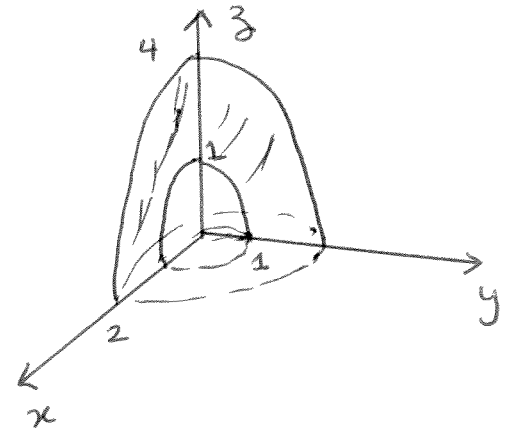
(14 points)

Let  $F(x, y, z) = xy^2\mathbf{i} + y^2\mathbf{j} + x^2\mathbf{k}$  be a field and  $S$  be the surface enclosing the volume between the paraboloids  $z = 4 - x^2 - y^2$  and  $z = 1 - x^2 - y^2$  and above the  $x - y$  axis.

Find the flux  $\iint_S F \cdot \mathbf{n} \, dS$ , where  $\mathbf{n}$  is the outer unit normal vector to  $S$ .

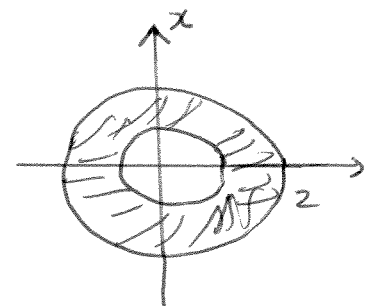
$$\text{flux} = \iint_S F \cdot \mathbf{n} \, dS$$

$$= \iiint_M \text{div} F \, dV = \iiint_M (y^2 + x^2) \, dV$$



Notice that  $M$  is the volume between two paraboloids so

$$\text{Vol}(M) = \text{Vol}(\text{parab. 1}) - \text{Vol}(\text{parab. 2})$$



$$\therefore \text{flux} = \iiint_{M_1} (y^2 + x^2) \, dV - \iiint_{M_2} (y^2 + x^2) \, dV$$

$$= \iint_{D_1} \int_0^{4-x^2-y^2} (y^2 + x^2) \, dz \, dA - \iint_{D_2} \int_0^{1-x^2-y^2} (y^2 + x^2) \, dz \, dA$$

where  $D_1$  and  $D_2$  are the larger and the smaller disk respectively

$$\therefore \text{flux} = \iint_{D_1} (x^2 + y^2) z \Big|_{z=0}^{z=4-x^2-y^2} \, dA - \iint_{D_2} (x^2 + y^2) z \Big|_{z=0}^{z=1-x^2-y^2} \, dA$$

$$\text{flux} = \iint_{D_1} \left[ y^2 z + \frac{3z^2}{2} \right]_0^{4-x^2-y^2} dA - \iint_{D_2} \left[ y^2 z + \frac{3z^2}{2} \right]_0^{1-x^2-y^2} dA$$

$$= \iint_{D_1} \left[ y^2(4-x^2-y^2) + \frac{3}{2}(4-x^2-y^2)^2 \right] dA - \iint_{D_2} \left[ y^2(1-x^2-y^2) + \frac{3}{2}(1-x^2-y^2)^2 \right] dA$$

use polar coordinates

$$\text{flux} = \int_0^{2\pi} \int_0^2 r^2 \sin^2 \theta (4$$

Question 5

(14 points)

a. Let  $f(z) = x^2 - x + y + i(y^2 - 5y - x)$ . Show that  $f(z)$  is differentiable at  $z_0 = 3 + 5i$  and find  $f'(z_0)$ .

b. If  $f(z) = e^x \cos y + iv(x, y)$  is an analytic function for any  $z$ .

Find  $v(x, y)$  and write  $f(z)$  in terms of  $z$ .

$$\text{Sol. } \frac{\partial u}{\partial x} = 2x - 1 \qquad \frac{\partial v}{\partial x} = -1$$

$$\frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial y} = 2y - 5$$

Since the partial derivatives are continuous then  $f$  is differentiable iff C-R-E hold.

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 1 \quad \text{holds}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Leftrightarrow 2x - 1 = 2y - 5 \Leftrightarrow \boxed{y = x + 2}$$

$z_0 = 3 + 5i$  satisfies  $y = x + 2$

Thus  $f$  is differentiable at  $z_0$ .

$$f'(z_0) = \frac{\partial u}{\partial x}(3, 5) + i \frac{\partial v}{\partial x}(3, 5) = 5 - i$$

b.  $f(z) = e^x \cos y + iv(x, y)$  is analytic  $\Rightarrow$  C-R-E hold

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x} = e^x \sin y \Rightarrow v = e^x \sin y + g(y)$$

$$\Rightarrow \frac{\partial v}{\partial y} = e^x \cos y + g'(y) = e^x \cos y$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C.$$

$$\text{Thus } v(x, y) = e^x \sin y + C$$

$$\Rightarrow f(z) = e^x \cos y + i e^x \sin y + iC$$

$$= e^x (\cos y + i \sin y) + \tilde{C}$$

$$= e^{x+iy} + \tilde{C} = e^z + \tilde{C}, \quad \tilde{C} \text{ is a constant.}$$

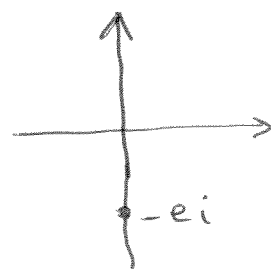
**Question 6**

(14 points)

a. If  $\ln(-ei) = a + ib$ , find  $a$  and  $b$ .b. **Solve:**  $2\cosh z + \sinh z = 2$ .c. **Show** that  $i\pi$  is a period for the periodic function  $f(z) = \tanh z$ .

Sol.  $\ln(-ei) = \log(-ei) = \ln|-ei| + i[\arg(-ei) + 2k\pi]$   
 $= \ln e + i\left(-\frac{\pi}{2} + 2k\pi\right)$   
 $= 1 + i\left(-\frac{\pi}{2} + 2k\pi\right)$

$a = 1$  and  $b = -\frac{\pi}{2} + 2k\pi$ ,  $k$  integer



b.  $2\cosh z + \sinh z = 2$

$e^z + e^{-z} + \frac{e^z - e^{-z}}{2} = 2$  (multiply by 2)

$2e^z + 2e^{-z} + e^z - e^{-z} = 4$

$3e^z + e^{-z} = 4$  (multiply by  $e^z$ )

$3e^{2z} - 4e^z + 1 = 0$

$(3e^z - 1)(e^z - 1) = 0$

$\Rightarrow e^z = \frac{1}{3}$  or  $e^z = 1$

$e^z = 1 \Rightarrow z = \ln 1 = \ln|1| + i(0 + 2k\pi) = 2ki\pi$

$e^z = \frac{1}{3} \Rightarrow z = \ln \frac{1}{3} + 2ki\pi$ .

So the solution set is

$\{2ki\pi, k \text{ integer}\} \cup \{\ln \frac{1}{3} + 2ki\pi, k \text{ integer}\}$ .

$$c. \tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1}$$

$$\tanh(z + i\pi) = \frac{e^{2z + 2i\pi} - 1}{e^{2z + 2i\pi} + 1} = \frac{e^{2z} - 1}{e^{2z} + 1} = \tanh z$$

Since  $e^z$  is periodic with a period  $2\pi i$

Question 7

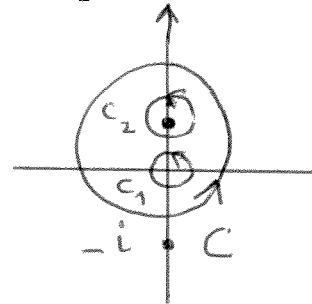
(14 points)

Use Cauchy's integral formulas to evaluate

$$I = \oint_C \frac{\sin z}{z^4 + z^2} dz, \text{ where } C \text{ the circle } |z - i| = \frac{3}{2}.$$

Sol.

$$I = \oint_C \frac{\sin z}{z^2(z^2 + 1)} dz$$



The singularities are  $0, \pm i$  but  $-i$  is outside  $C$ .

So

$$I = \oint_{C_1} \frac{\sin z}{z^2} dz + \oint_{C_2} \frac{\sin z}{z^2(z+i)} dz$$

$$= 2\pi i \left. \frac{d}{dz} \left( \frac{\sin z}{z^2 + 1} \right) \right|_{z=0} + 2\pi i \left. \frac{\sin z}{z^2(z+i)} \right|_{z=i}$$

$$= 2\pi i \left. \frac{(z^2 + 1) \cos z - 2z \sin z}{(z^2 + 1)^2} \right|_{z=0} + 2\pi i \frac{\sin i}{-2i}$$

$$= 2\pi i (1) - \pi \sin i = 2\pi i - \pi \sin i$$





Question 8

(14 points)

Let  $f(z) = \frac{e^{-iz^2} - 1}{z^5}$ .

a. Find the Laurent series in  $|z| > 0$  and classify its pole.

b. Use the result in (a) to find  $\oint_C f(z) dz$ , where C is given in the adjacent figure.

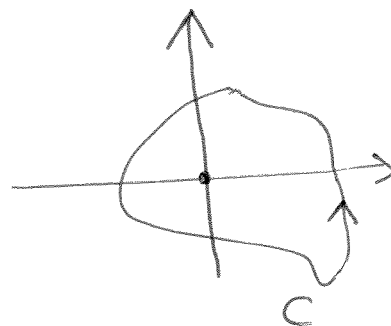
a.  $e^{-iz^2} = \sum_{k=0}^{\infty} \frac{(-iz^2)^k}{k!}$

$\Rightarrow e^{-iz^2} - 1 = \sum_{k=1}^{\infty} \frac{(-i)^k}{k!} z^{2k}$

$\Rightarrow \frac{e^{-iz^2} - 1}{z^5} = \sum_{k=1}^{\infty} \frac{(-i)^k}{k!} z^{2k-5}$

b.

$f(z) = -iz^{-3} - \frac{1}{2}z^{-1} + \frac{i}{6}z - \dots$



$\oint_C f(z) dz = 2\pi i C_{-1} = -\pi i$



Question 9

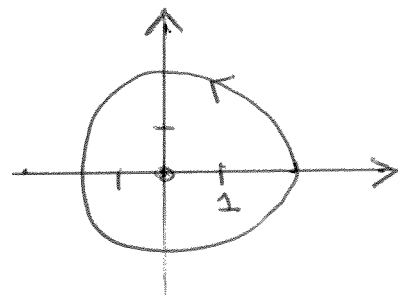
(14 points)

Use the residue theorem to evaluate

$$\oint_C \frac{e^z - 1}{z^2(z-1)^2} dz, \text{ where } C \text{ is the circle } |z| = 2.$$

$$f(z) = \frac{e^z - 1}{z^2(z-1)^2}$$

0, 1 are singularities



$$\bullet \lim_{z \rightarrow 0} z^2 f(z) = \lim_{z \rightarrow 0} \frac{e^z - 1}{(z-1)^2} = 0$$

$$\bullet \lim_{z \rightarrow 0} z f(z) = \lim_{z \rightarrow 0} \frac{e^z - 1}{z(z-1)^2} = \frac{0}{0} \text{ (L'Hopital)}$$

$$= \lim_{z \rightarrow 0} \frac{e^z}{(z-1)^2 + 2z(z-1)} = \frac{1}{1} = 1 \neq 0$$

So 0 is a simple pole and  $\text{Res}(f, 0) = 1$ .

$$\bullet \lim_{z \rightarrow 1} (z-1)^2 f(z) = \frac{e-1}{1} = e-1 \neq 0$$

Thus 1 is a pole of order 2

$$\text{Res}(f, 1) = \frac{1}{1!} \lim_{z \rightarrow 1} \frac{d}{dz} \left( (z-1)^2 f(z) \right)$$

$$= \lim_{z \rightarrow 1} \frac{d}{dz} \left( \frac{e^z - 1}{z^2} \right) = \lim_{z \rightarrow 1} \frac{e^z z^2 - 2z(e^z - 1)}{z^4}$$

$$= \frac{e - 2(e-1)}{1} = 2 - e.$$

$$\oint_C f(z) dz = 2\pi i [\text{Res}(f, 0) + \text{Res}(f, 1)] = 2\pi i [3 - e].$$



## Question 10

(14 points)

Compute  $\int_{-\infty}^{+\infty} \frac{2x^2-1}{x^4+5x^2+4} dx$ .

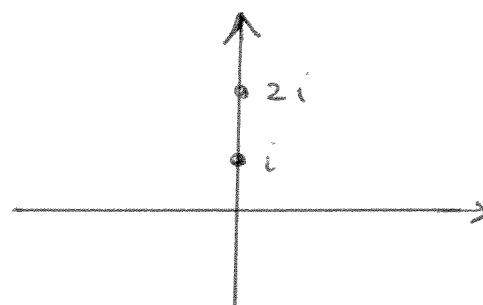
$$\varphi(z) = z^4 + 5z^2 + 4 = (z^2 + 1)(z^2 + 4)$$

So  $f(z) = \frac{2z^2 - 1}{(z^2 + 1)(z^2 + 4)}$  has  $i, 2i$  as poles

in the upper half plane

Since  $d\varphi - dp = 4 - 2 = 2$

and  $\varphi(x) \neq 0, \forall x \in \mathbb{R}$



then

$$\int_{-\infty}^{\infty} f(x) dx = 2\pi i [\text{Res}(f, i) + \text{Res}(f, 2i)]$$

$$\lim_{z \rightarrow i} (z - i) f(z) = \lim_{z \rightarrow i} \frac{2z^2 - 1}{(z + i)(z^2 + 4)} = \frac{-3}{6i} = -\frac{1}{2i}$$

$\Rightarrow i$  is a simple pole and  $\text{Res}(f, i) = -\frac{1}{2i}$

$$\lim_{z \rightarrow 2i} (z - 2i) f(z) = \lim_{z \rightarrow 2i} \frac{2z^2 - 1}{(z^2 + 1)(z + 2i)} = \frac{-9}{(-3)(4i)} = \frac{3}{4i}$$

$$\therefore \int_{-\infty}^{\infty} f(z) dz = 2\pi i \left[ -\frac{1}{2i} + \frac{3}{4i} \right] = 2\pi \left( \frac{3-2}{4} \right) = \frac{2\pi}{4} = \frac{\pi}{2}$$

