King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 302 Final Exam

Semester (141) Dec. 30, 2014 at 8:00-11:00 AM

Name: ....

I.D: .... Section: .... Serial: ....

Question	Points
1	/14
2	/14
3	/14
4	/14
5	/14
6	/14
7	/14
8	/14
9	/14
10	/14
Total	/140

Question 1

Question 1 (14 points)

a. Let $A = \begin{pmatrix} -1 & 3 & 0 \\ 3 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, use the Guaussian elimination method to **find** A^{-1}.

b. Use (a) to **solve** the system

$$\begin{cases} -x_1 + 3x_2 = 1 \\ 3x_1 - 2x_2 + x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases}$$

Sol.

a. $\hat{A} = \left[\begin{array}{ccc|ccc} -1 & 3 & 0 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 + 3R_1]{-R_1}$

$\left[\begin{array}{ccc|ccc} 1 & -3 & 0 & -1 & 0 & 0 \\ 0 & 7 & 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$

$\left[\begin{array}{ccc|ccc} 1 & -3 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 7 & 1 & 3 & 1 & 0 \end{array}\right] \xrightarrow[R_3 - 7R_2]{R_1 + 3R_2}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 6 & -1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & -13 & 3 & 1 & -7 \end{array}\right] \xrightarrow{R_3 / -13}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 6 & -1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3/13 & -1/13 & 7/13 \end{array}\right] \xrightarrow[R_2 - 2R_3]{R_1 - 6R_3}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/13 & 6/13 & -3/13 \\ 0 & 1 & 0 & 6/13 & 2/13 & -1/13 \\ 0 & 0 & 1 & -3/13 & -1/13 & 7/13 \end{array}\right]$

So $A^{-1} = \begin{bmatrix} 5/13 & 6/13 & -3/13 \\ 6/13 & 2/13 & -1/13 \\ -3/13 & -1/13 & 7/13 \end{bmatrix}$

b)

b) we have $AX = b$.

So $X = A^{-1} b$

$$= \begin{bmatrix} \frac{5}{13} & \frac{6}{13} & -\frac{3}{13} \\ \frac{6}{13} & 2/13 & -1/13 \\ -\frac{3}{13} & -\frac{1}{13} & 7/13 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5/13 \\ 6/13 \\ -3/13 \end{bmatrix}.$$

Question 2 (14 points)

Let $A = \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix}$

a) Find all **eigenvalues** of A and the corresponding **eigenvectors**.

b) **Find** a matrix P that diagonalizes A.

c) If $D = P^{-1}AP$, **find** D^4 and write a **formula** to evaluate A^4. (Do not calculate A^4)

Sol. a.

$$\begin{vmatrix} 1-\lambda & 0 & 7 \\ 0 & 1-\lambda & 0 \\ 7 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} + 7 \begin{vmatrix} 0 & 1-\lambda \\ 7 & 0 \end{vmatrix}$$

$$= (1-\lambda)^3 - 49(1-\lambda)$$

$$= (1-\lambda)\left[(\lambda-1)^2 - 49\right]$$

$$= (1-\lambda)\left[(\lambda-8)(\lambda+6)\right] = 0$$

$$\Rightarrow \lambda = 1, -6, 8 \text{ are eigenvalues.}$$

Eigenvectors

$\boxed{\lambda = -6}$ $\begin{pmatrix} 7 & 0 & 7 \\ 0 & 7 & 0 \\ 7 & 0 & 7 \end{pmatrix} \xrightarrow[\substack{R_3 - R_1 \\ R_2/7}]{R_1/7} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow x_1 = -x_3 , \quad x_2 = 0$$

So $E_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is an eigenvector.

$\boxed{\lambda = 1}$ $\begin{pmatrix} 0 & 0 & 7 \\ 0 & 0 & 0 \\ 7 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 7 & 0 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow x_1 = x_3 = 0 \Rightarrow E_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ is an eigenvector.}$$

$\boxed{\lambda = 8}$ $\begin{pmatrix} -7 & 0 & 7 \\ 0 & -7 & 0 \\ 7 & 0 & -7 \end{pmatrix} \xrightarrow[\substack{R_2/-7 \\ R_3 + R_1}]{R_1/-7} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow x_2 = 0 , \quad x_1 = x_3 \Rightarrow E_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ is an eigenvector}$$

b. $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ diagonalizes A such that

$$P^{-1} A P = D = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

c. $D^4 = \begin{pmatrix} 6^4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8^4 \end{pmatrix}$
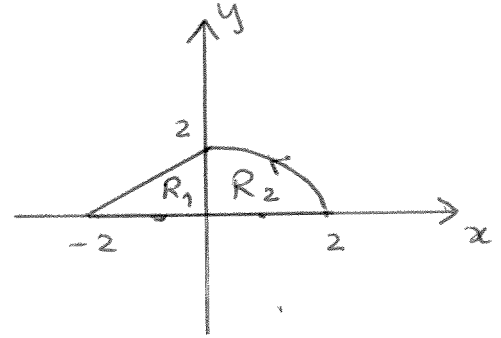
$$A^4 = P D^4 P^{-1}$$

Find $\oint_C (-xy)dx + (e^{-y^2})dy,$ where C is the positively oriented closed path consisting of the quarter circle joining the points $(2,0), (0,2),$ the line segment joining the points $(0,2), (-2,0),$ and the line segment joining the points $(-2,0), (2,0).$

<u>sol.</u>

$$\oint_C (-xy)dx + e^{-y^2}dy$$

$$= \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dA = \iint_R x\,dA$$

$$= \iint_{R_1} x\,dA + \iint_{R_2} x\,dA$$

$$= \int_{-2}^{0}\int_{0}^{x+2} x\,dy\,dx + \int_{0}^{\pi/2}\int_{0}^{2}(r\cos\theta)r\,dr\,d\theta$$

$$= \int_{-2}^{0} x(x+2)dx + \int_{0}^{\pi/2}\cos\theta\,d\theta\int_{0}^{2}r^2\,dr$$

$$= \frac{x^3}{3} + x^2\Big]_{-2}^{0} + \sin\theta\Big|_{0}^{\pi/2}\;\frac{r^3}{3}\Big|_{0}^{2}$$

$$= \frac{8}{3} - 4 + (1)\left(\frac{8}{3}\right) = \frac{16}{3} - \frac{12}{3} = \frac{4}{3}.$$

$$= \iint_{D_1} (x^2 + y^2)(4 - x^2 - y^2)\, dA - \iint_{D_2} (x^2 + y^2)(1 - x^2 - y^2)\, dA$$

We then use polar coordinates:

$$\text{flux} = \int_0^{2\pi} \int_0^2 r^2(4 - r^2)\, r\, dr\, d\theta - \int_0^{2\pi} \int_0^1 r^2(1 - r^2)\, r\, dr\, d\theta$$

$$= 2\pi \int_0^2 (4r^3 - r^5)\, dr - 2\pi \int_0^1 (r^3 - r^5)\, dr$$

$$= 2\pi \left(r^4 - \frac{r^6}{6} \right)\Big|_0^2 - 2\pi \left[\frac{r^4}{4} - \frac{r^6}{6} \right]_0^1$$

$$= 2\pi \left(16 - \frac{64}{6} \right) - 2\pi \left(\frac{1}{4} - \frac{1}{6} \right)$$

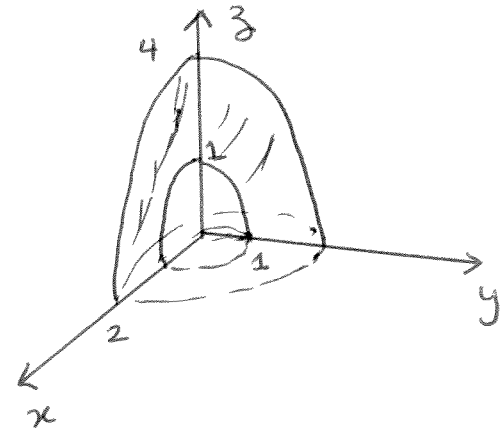$$= 2\pi \, \frac{32}{6} - \frac{2\pi}{12} = \frac{32\pi}{3} - \frac{\pi}{6} = \frac{21\pi}{2}.$$

Let $\mathbf{F}(x,y,z) = xy^2\mathbf{i} + y\overset{2}{2}\mathbf{j} + x\overset{be}{}\mathbf{k}$ be a field and S is the surface enclosing the

volume between the paraboloids $z = 4 - x^2 - y^2$ and $z = 1 - x^2 - y^2$ and

above the $x - $ axis.

Find the flux $\iint_S \mathbf{F}.\mathbf{n}\,dS$, where $\mathbf{n}$ is the outer unit normal vector to S.
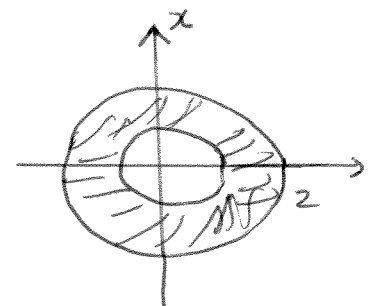
$$\text{flux} = \iint_S \mathbf{F}.\mathbf{n}\,dS$$

$$= \iiint_M \text{div}\, F\, dv = \iiint_M (y^2 + x^2)\cdot dv$$

Notice that M is the volume between
two paraboloids So

$$\text{Vol}(M) = \text{Vol}(\text{parab. 1}) - \text{Vol}(\text{parab 2})$$

$$\therefore \text{flux} = \iiint_{M_1}(y^2 + x^2)\,dv - \iiint_{M_2}(y^2 + x^2)\,dv$$

Surface below

$$= \iint_{D_1}\int_0^{4-x^2-y^2}(y^2 + x^2)\,dz\,dA - \iint_{D_2}\int_0^{1-x^2-y^2}(y^2 + x^2)\,dz\,dA$$

where D_1 and D_2 are the larger and the smaller disk
respectively

$$\therefore \text{flux} = \iint_{D_1}(x^2+y^2)\,z\Big|_{z=0}^{z=4-x^2-y^2}\,dA - \iint_{D_2}(x^2+y^2)\,z\Big|_0^{1-x^2-y^2}\,dA$$

$$\text{flux} = \iint\limits_{D_1} \left[y^2 z + \frac{3}{2}\frac{z^2}{2} \right]_0^{4-x^2-y^2} dA - \iint\limits_{D_2} \left(y^2 z + \frac{3z^2}{2} \right)\Big|_0^{1-x^2-y^2} dA$$

$$= \iint\limits_{D_1} \left[y^2(4-x^2-y^2) + \frac{3}{2}(4-x^2-y^2)^2 \right] dA - \iint\limits_{D_2} \left[y^2(1-x^2-y^2) + \frac{3}{2}(1-x^2-y^2)^2 \right] dA$$

use polar coordinates

$$\text{flux} = \int_0^{2\pi} \int_0^2 r^2 \sin^2\theta\,(4$$

a. Let $f(z) = x^2 - x + y + i(y^2 - 5y - x)$. **Show** that $f(z)$ is differentiable at

   $z_0 = 3 + 5i$ and **find** $f'(z_0)$.

b. If $f(z) = e^x \cos y + iv(x, y)$ is an analytic function for any z.

   Find $v(x, y)$ and **write** $f(z)$ in terms of z.

$\underline{Sol.}$ $\underline{a.}$

$\dfrac{\partial u}{\partial x} = 2x - 1$ $\qquad\qquad\qquad \dfrac{\partial v}{\partial x} = -1$

$\dfrac{\partial u}{\partial y} = 1$ $\qquad\qquad\qquad \dfrac{\partial v}{\partial y} = 2y - 5$

Since the partial derivatives are continuous then f is

differentiable iff C_-R_-E hold.

$\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} = 1$ $\qquad$ holds

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \iff 2x - 1 = 2y - 5 \iff \boxed{y = x + 2}$

$z_0 = 3 + 5i$ satisfies $y = x + 2$

Thus f is differentiable at z_0

$f'(z_0) = \dfrac{\partial u}{\partial x}(3, 5) + i\dfrac{\partial v}{\partial x}(3, 5) = 5 - i$

$\underline{b.}$ $\quad f(z) = e^x \cos y + iv(x, y)$ is analytic $\implies C_-R_-E$ hold

$\dfrac{\partial u}{\partial x} = e^x \cos y = \dfrac{\partial v}{\partial y}$

$\dfrac{\partial u}{\partial y} = -e^x \sin y = -\dfrac{\partial v}{\partial x}$

$\dfrac{\partial v}{\partial x} = e^x \sin y \implies v = e^x \sin y + g(y)$

$\implies \dfrac{\partial v}{\partial y} = e^x \cos y + g'(y) = e^x \cos y$

$\Rightarrow \quad g'(y) = 0 \quad \Rightarrow \quad g(y) = c.$

Thus $\quad v(x,y) = e^{x} \sin y + c$

$\Rightarrow \quad f(z) = e^{x} \cos y + i\, e^{x} \sin y + ic$

$\qquad = e^{x} (\cos y + i \sin y) + \hat{c}$

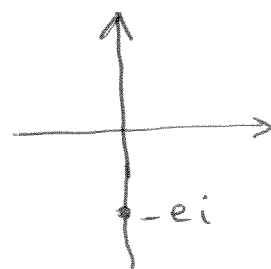$\qquad = e^{x+iy} + \hat{c} \; = e^{z} + \hat{c}, \quad \hat{c}$ is a constant.

a. If $\ln(-ei) = a + ib$, **find** a and b.

b. **Solve:** $2\cosh z + \sinh z = 2$.

c. **Show** that $i\pi$ is a period for the periodic function $f(z) = \tanh z$.

Sol. $\ln(-ei) = \log(-ei) = \ln|-ei| + i\left[\arg(-ei) + 2k\pi\right]$

$$= \ln e + i\left(-\frac{\pi}{2} + 2k\pi\right)$$

$$= 1 + i\left(-\frac{\pi}{2} + 2k\pi\right)$$

$a = 1$ and $b = -\frac{\pi}{2} + 2k\pi$, k integer

$-ei$

b. $2\cosh z + \sinh z = 2$

$$e^z + e^{-z} + \frac{e^z - e^{-z}}{2} = 2 \qquad (\text{multiply by } 2)$$

$$2e^z + 2e^{-z} + e^z - e^{-z} = 4$$

$$3e^z + e^{-z} = 4 \qquad \left(\text{Multiply by } e^z\right)$$

$$3e^{2z} - 4e^z + 1 = 0$$

$$\left(3e^z - 1\right)\left(e^z - 1\right) = 0$$

$$\Rightarrow e^z = \frac{1}{3} \quad \text{or} \quad e^z = 1$$

$e^z = 1 \Rightarrow z = \ln 1 = \ln|1| + i(0 + 2k\pi) = 2ki\pi$

$e^z = \frac{1}{3} \Rightarrow z = \ln \frac{1}{3} + 2ki\pi$.

So the solution set is

$$\left\{2ki\pi, \ k \text{ integer}\right\} \cup \left\{\ln\frac{1}{3} + 2ki\pi, \ k \text{ integer}\right\}.$$

c. $\tanh z = \dfrac{\sinh z}{\cosh z} = \dfrac{e^{z} - e^{-z}}{e^{z} + e^{-z}} = \dfrac{e^{2z} - 1}{e^{2z} + 1}$

$\tanh(z + i\pi) = \dfrac{e^{2z + 2i\pi} - 1}{e^{2z + 2i\pi} + 1} = \dfrac{e^{2z} - 1}{e^{2z} + 1} = \tanh z$
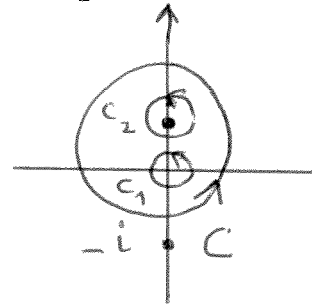
Since e^{z} is periodic with a period $2\pi i$

Use Cauchy's integral formulas to **evaluate**

$$I = \oint_C \frac{\sin z}{z^4 + z^2}\, dz \text{, where } C \text{ the circle } |z - i| = \frac{3}{2}.$$

is

<u>Sol.</u>

$$I = \oint_C \frac{\sin z}{z^2(z^2 + 1)}\, dz$$

The singularities are $0, \pm i$ but $-i$ is outside C.

So

$$I = \oint_{C_1} \frac{\frac{\sin z}{z^2 + 1}}{z^2}\, dz + \oint_{C_2} \frac{\frac{\sin z}{z^2(z+i)}}{z - i}\, dz$$

$$= 2\pi i \left. \frac{d}{dz}\left(\frac{\sin z}{z^2 + 1}\right)\right|_{z=0} + 2\pi i \left. \frac{\sin z}{z^2(z+i)}\right|_{z=i}$$

$$= 2\pi i \left. \frac{(z^2 + 1)\cos z - 2z \sin z}{(z^2 + 1)^2}\right|_{z=0} + 2\pi i \frac{\sin i}{-2i}$$

$$= 2\pi i (1) - \pi \sin i = 2\pi i - \pi \sin i$$

Let $f(z) = \dfrac{e^{-iz^2} - 1}{z^5}$.

a. **Find** the Laurent series in $|z| > 0$ and **classify** its pole.

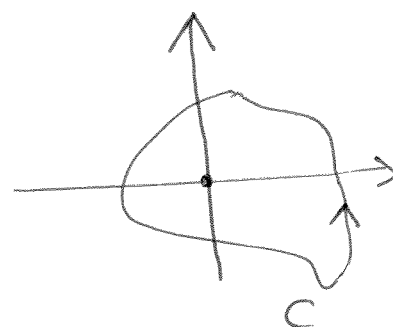b. Use the result in (a) to **find** $\oint_C f(z)\,dz$, where C is given in the adjacent figure.

a. $\quad e^{-iz^2} = \sum\limits_{k=0}^{\infty} \dfrac{(-iz^2)^k}{k!}$

$\Rightarrow \quad e^{-iz^2} - 1 = \sum\limits_{k=1}^{\infty} \dfrac{(-i)^k}{k!} z^{2k}$

$\Rightarrow \quad \dfrac{e^{-iz^2} - 1}{z^5} = \sum\limits_{k=1}^{\infty} \dfrac{(-i)^k}{k!} z^{2k-5}$

b.

$f(z) = -i\, z^{-3} - \dfrac{1}{2} z^{-1} + \dfrac{i}{6} z - \cdots$
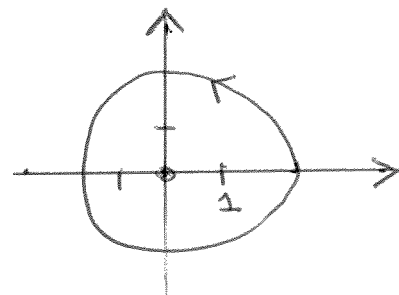

C

$\oint_C f(z)\,dz = 2\pi i\, C_{-1} = -\pi i$

Use the residue theorem to evaluate

$$\oint_C \frac{e^z - 1}{z^2(z-1)^2} \, dz, \text{ where } C \text{ is the circle } |z| = 2.$$

$$f(z) = \frac{e^z - 1}{z^2(z-1)^2}$$

$0, 1$ are singularities

- $\lim\limits_{z \to 0} z^2 f(z) = \lim\limits_{z \to 0} \frac{e^z - 1}{(z-1)^2} = 0$

- $\lim\limits_{z \to 0} z f(z) = \lim\limits_{z \to 0} \frac{e^z - 1}{z(z - 1)^2} = \frac{0}{0}$ (L'Hopital)

$$= \lim\limits_{z \to 0} \frac{e^z}{(z-1)^2 + 2z(z-1)} = \frac{1}{1} = 1 \neq 0$$

So 0 is a simple pole and $\text{Res}(f, 0) = 1$.

- $\lim\limits_{z \to 1} (z-1)^2 f(z) = \frac{e - 1}{1} = e - 1 \neq 0$

Thus 1 is a pole of order 2

$$\text{Res}(f, 1) = \frac{1}{1!} \lim\limits_{z \to 1} \frac{d}{dz} \left((z-1)^2 f(z) \right)$$

$$= \lim\limits_{z \to 1} \frac{d}{dz} \left(\frac{e^z - 1}{z^2} \right) = \lim\limits_{z \to 1} \frac{e^z z^2 - 2z(e^z - 1)}{z^4}$$

$$= \frac{e - 2(e - 1)}{1} = 2 - e.$$

$$\oint_C f(z) \, dz = 2\pi i \left[\text{Res}(f, 0) + \text{Res}(f, 1) \right] = 2\pi i [3 - e].$$

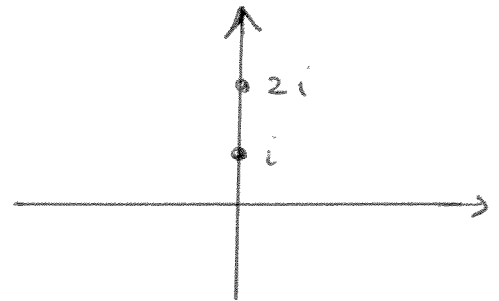**Compute** $\int_{-\infty}^{+\infty} \frac{2x^2-1}{x^4+5x^2+4} dx$.

$\varphi(z) = z^4 + 5z^2 + 4 = (z^2+1)(z^2+4)$

So $f(z) = \dfrac{2z^2-1}{(z^2+1)(z^2+4)}$ has $i, 2i$ as poles

in the upper half plane

Since $d\varphi - dp = 4 - 2 = 2$

and $\varphi(x) \neq 0, \forall x \in \mathbb{R}$

then

$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \left[Res(f,i) + Res(f,2i) \right]$

$\lim_{z \to i} (z-i)f(z) = \lim_{z \to i} \dfrac{2z^2-1}{(z+i)(z^2+4)} = \dfrac{-3}{6i} = -\dfrac{1}{2i}$

$\Rightarrow i$ is a simple pole and $Res(f,i) = -\dfrac{1}{2i}$

$\lim_{z \to 2i} (z-2i)f(z) = \lim_{z \to 2i} \dfrac{2z^2-1}{(z^2+1)(z+2i)} = \dfrac{-9}{(-3)(4i)} = \dfrac{3}{4i}$

$\therefore \int_{-\infty}^{\infty} f(z)dz = 2\pi i \left[-\dfrac{1}{2i} + \dfrac{3}{4i} \right] = 2\pi \left(\dfrac{3-2}{4} \right) = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$.